Projective dynamics and first integrals

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Abstract. We show the relation between Appell's remark on the central projection in the dynamic of a particle and T.-Y. Thomas and Nijenhuis studies of the polynomial first integrals of the geodesic flow on a space of constant curvature. Among the consequences: the space of leading terms of a quadratic first integral of an equation $\ddot{q} = f(q)$, $q \in \mathbb{R}^n$, is isomorphic to the space of quadrilinear forms on $\mathbb{R}^{n+1}$ having the symmetries of the Riemannian curvature tensor. Such a leading term also appears as a quadratic form in what we call the projective impulsion bivector. We characterize the cases where a quadratic first integral comes from a Lagrangian, thus giving a geometrical interpretation and a converse to some recent results by Lundmark.

1. Introduction

To explain what we call projective dynamics, we will explain the relation between the dynamics of the so-called natural systems and projective dynamics. We compare it with the intuitive relation between the Euclidean space and the real projective space of the same dimension. The transformation of the first into the second has a global aspect and a local aspect. At the global or topological level we "add" something at infinity, namely a projective plane. This gluing compactifies the space. At the local level we "forget" some structure. We first forget the Euclidean form to get the so-called affine space. Then we forget something else. We can say for example we forget the standard parallel transport but still keep in mind a special class of mutually indistinguishable parallel transports (each one corresponding to an affine chart).

Changing the affine space into the projective space we lose the standard parametrizations of the lines. In other words we lose the concept of uniform rectilinear motion of a point. This dynamical statement is one of the keys of projective dynamics. It also relates the global aspect to the local aspect: the concept of uniform motion does not fit with the projective line, because the point at infinity is never reached and the projective line is never described completely.

A natural dynamical system is described by an equation of the form $\ddot{q} = \nabla U(q)$, where q is the position of a particle in a Euclidean space, U is a function called the force function, and ∇U denotes the gradient of U. This equation can also model a multiparticle system. The position space, where q lives, is then the Cartesian product of the position spaces of each particle.

Natural dynamics corresponds to Euclidean geometry, and we can call it Euclidean dynamics. The Euclidean form is used to define the gradient, and is a fundamental ingredient of the first integral of energy. If we forget the Euclidean

structure of the position space, we enter a larger and more basic class of systems, those of the form $\ddot{q} = f(q)$, where $f(q)$ is a field of forces depending only on the position. They are called Newton systems in [Lun], and for us they pertain to affine dynamics. It is worth to mention that Jacobi [Jac] introduced what he called the last multiplier in the context of this class of *a priori* non-Hamiltonian systems.

Our main claim is that there exists a projective dynamics, i.e. a dynamics where the position space is the real projective space, or its double covering, or more frequently an open subset of this double covering. We gave a description in [Alb] and we will give a very elementary explanation in the next section. The local aspects of projective dynamics will be presented in an indirect way: to forget some local structure, we forget the "screen" and thus consider as equivalent many affine or "curved-affine" dynamical systems.

The global aspect of projective dynamics is the possible prolongation "after infinity" of the orbits in a natural system. The first example is the equation $\ddot{q} = 0$, where the rectilinear trajectory is prolongated in the projective line. For the two body problem the prolongation is as expected: the hyperbolic trajectories continue and describe a complete ellipse of the projective space. We showed in [Alb] that there is also a prolongation for the motion of a particle in the gravitational field of any fixed repartition of mass. For the three body problem, the possibility of a prolongation was discussed by Chazy [Cha], who obtained a rather negative answer. Only some hyperbolic trajectories have a natural prolongation after crossing infinity. In our language the projective force field becomes singular at infinity. It is interesting to note that if we define the interaction between the bodies by an inverse cube law instead of the Newtonian inverse square law, this singularity disappears. There is a prolongation for all the unbounded orbits.

2. Systems defined by a field of forces

Halphen and Appell discovered the properties of the central projection in dynamics (see [Alb] for the references). They considered mostly the central projection from an affine plane in $\mathbb{R}^3$, where a particle moves, to another plane where the projected particle moves. But Appell in a remark mentioned the possibility of projecting on a sphere or on the two-sheeted hyperboloid in Minkowski space, in order to get the remarkable Kepler problem in a constant curvature space. We will first put Appell's transformation at work on a larger class of systems.

2.1. Let V be a real vector space of finite dimension $n+1$. A set $\Omega \subset V \setminus \{0\}$ is said *semi-conic* if, for any $q \in \Omega$ and any $\lambda > 0$, $\lambda q \in \Omega$. We consider a semi-conic connected open set Ω together with

(i) a smooth function $h : \Omega \to \mathbb{R}$ with the property $h(\lambda q) = \lambda h(q)$ for any $q \in \Omega$, $\lambda > 0$, which defines the *screen* $\mathcal{H} = \{q \in \Omega | h(q) = 1\}$,

(ii) a smooth "force field depending only on the position" $f : \mathcal{H} \to V$, tangent to $\mathcal{H}$.

In words h is *positively homogeneous* of degree one. The data (i) and (ii) defines

a differential system that we write

$$\ddot{q} = f + \lambda q. \tag{2.1}$$

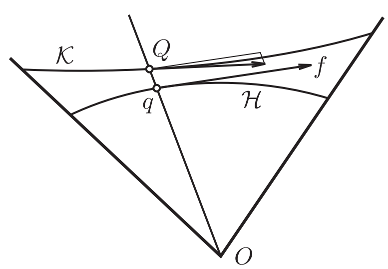
The term λq is the *radial reaction*. The factor $\lambda \in \mathbb{R}$ is uniquely determined at any time in such a way that the particle remains on the screen $\mathcal{H}$. By this we mean that the initial condition $(q, \dot{q})$ being in $T\mathcal{H} = \{(q, \dot{q}) \in \Omega \times V | h(q) = 1, dh|_q(\dot{q}) = 0\}$, the unique solution continues on $T\mathcal{H}$.

To see that (2.1) is a well-defined system of ordinary differential equations, we observe that λ is determined in a unique way as a function of $(q, \dot{q})$, as follows. By successive derivations of the condition $h(q) = 1$ we find $dh|_q(\dot{q}) = 0$ and $\partial^2 h|_q(\dot{q}, \dot{q}) + dh|_q(\ddot{q}) = 0$. The tangency hypothesis in (ii) is $dh|_q(f) = 0$. By Euler's relation $dh|_q(q) = h(q) = 1$, thus $\lambda(q, \dot{q}) = -\partial^2 h|_q(\dot{q}, \dot{q})$. Substituting this expression in (2.1) and arbitrarily extending f to Ω we get an ordinary system of differential equations on $\Omega \times V$ which possesses $T\mathcal{H}$ as a $2n$-dimensional invariant submanifold.

2.2. *Change of time.* We denote as well by d/dt the above dot derivative. We introduce another time parameter s. We denote d/ds by $'$. The change of time is defined by $a = ds/dt$. For any variable x we have $\dot{x} = dx/dt = adx/ds = ax'$. Equation (2.1) is changed into

$$a(aq')' = f|_q + \lambda q. \tag{2.2}$$

2.3. *Central projection.* We consider an equation $Q'' = g|_Q + \mu Q$ of the same form as (2.1), but defined on a different screen $\mathcal{K}$ with equation $k(Q) = 1$ by a force field g. We project centrally Q on $\mathcal{H}$, setting $Q = bq$.



The change of screen is defined by the function $b(q) = h(q)/k(q)$, positively homogeneous of degree 0 on Ω. We consider the motion of $q \in \mathcal{H}$ defined by the motion of $Q \in \mathcal{K}$ according to $Q'' = g|_Q + \mu Q$. Then q satisfies

$$(bq)'' = g|_{bq} + \mu bq. \tag{2.3}$$

2.4. *Change of time and change of screen.* Expanding and normalizing the q'' term we compare the change of time with the change of screen:

$$q'' + \frac{a'}{a}q' = \frac{1}{a^2}f|_q + \frac{\lambda}{a^2}q, \qquad q'' + 2\frac{b'}{b}q' = \frac{1}{b}g|_{bq} + (\mu - \frac{b''}{b})q. \tag{2.4}$$

If we make $a = b^2$ and $g|_{bq} = b^{-3} f|_q$ the first three terms coincide. As the fourth term is determined uniquely by the condition $q \in \mathcal{H}$, it is the same in both equations. Both equations define the same motion of q on the screen $\mathcal{H}$.

An objection is that the field of force $g|_{bq} = b^{-3} f|_q$ is not tangent to the screen $\mathcal{K}$ at the point $Q = bq$. Axiom (ii) is not respected. This objection is easily removed. The factor λ in Equation (2.1) is defined in order to maintain the constraint $h(q) = 1$. If f is not tangent to the screen, we change the formula $\lambda(q, \dot q) = -\partial^2 h|_q(\dot q, \dot q)$ into the more general formula $\lambda(q, \dot q) = -\partial^2 h|_q(\dot q, \dot q) - dh|_q(f|_q)$. Now a force field f is equivalent to a force field $f + \gamma q$, with any $\gamma : \Omega \to \mathbb{R}$. We call γq a *radial vector field*. There is a unique choice of radial vector field that makes the force tangent to the screen. But when we find it convenient we work with forces which are not tangent to the screen.

If we extend f, previously defined on $\mathcal{H}$, in

(iii) a smooth vector field $f : \Omega \to V$, positively homogeneous of degree -3,

the solutions of $Q'' = f|_Q + \mu Q$ on $\mathcal{K}$ are sent on the solutions of $\ddot q = f|_q + \lambda q$ on $\mathcal{H}$ by central projection and change of time.

A vector field f satisfying Axiom (iii), and defined up to the addition of a radial vector field, is called a *projective field of force*. The founding assertion of *projective dynamics* is that the data (iii) can substitute the data (i) and (ii) of the previous section. The screen defined by (i) appears as unessential as soon as we remark that another screen defines the same dynamics, except for the time parametrization. Let us rephrase our result.

2.5. *Definition.* The *restriction* of the projective force field $f : \Omega \to V$ to the screen $\mathcal{H}$, with equation $h(q) = 1$, is the vector field $f_h : \mathcal{H} \to V$, tangent to $\mathcal{H}$, with expression $f_h = dh \lrcorner (q \wedge f) = f - dh(f)q$.

2.6. *Definition.* Given two screens $\mathcal{H}$ and $\mathcal{K}$, the *extended central projection* is the application $T\mathcal{H} \to T\mathcal{K}$, $(q, \dot q) \mapsto (Q, Q')$, such that $Q = bq$ for some $b > 0$ and $q \wedge \dot q = Q \wedge Q'$.

2.7. *Founding statement.* A projective force field f, restricted to two screens $\mathcal{H}$ and $\mathcal{K}$ defines on each screen a tangent force field, i.e. a differential system of type (2.1). The extended central projection from $\mathcal{H}$ to $\mathcal{K}$ sends trajectory on trajectory, velocity vector on velocity vector, without respecting the time parameters.

The example treated by Appell is the projection of the usual Kepler problem on the Euclidean space to the Kepler-Serret-Killing problems on the spaces of constant curvature. In the first problem the screen function h is a linear form on V. In the second it is the square root of a non-degenerate quadratic form. Appell did not point out the fundamental invariant $q \wedge \dot q = Q \wedge Q'$, that we called in [Alb] the *projective impulsion bivector*.

3. Homogeneous form of the first integrals

3.1. Any force field (ii) possesses a homogeneous form, the projective force field (iii). Any function $G_{\mathcal{H}}$ defined on $T\mathcal{H} = \{(q, v) \in \Omega \times V | h(q) = 1, \langle dh, v\rangle = 0\}$

has a unique homogeneous form

$$G(q,v)=G_{\mathcal{H}}\Big(\frac{q}{h(q)},\langle dh,q\rangle v-\langle dh,v\rangle q\Big). \tag{3.1}$$

The function G is defined on $\Omega\times V$ and coincides with $G_{\mathcal{H}}$ on $T\mathcal{H}$. It satisfies

$$G(q,v+\gamma q)=G(q,v),\qquad G(\lambda q,\lambda^{-1}v)=G(q,v) \tag{3.2}$$

for any $\gamma\in\mathbb{R}$ and any $\lambda>0$. If an extended central projection maps $(q,\dot q)$ to (Q,Q'), then $G(q,\dot q)=G(Q,Q')$. *If $G_{\mathcal{H}}$ is a first integral of System* (2.1), *if we homogenize f in a projective force field, the dynamics* (2.1) *obtained by restriction to any screen possesses G as a first integral.*

Let $G(q,v)$ be a function satisfying (3.2). Consider the time derivative of $G(q,\dot q)$ along a solution of $\ddot q=f+\lambda q$. It defines new function $\dot G(q,v)=\langle\partial G/\partial q,v\rangle+\langle\partial G/\partial v,f\rangle$. The radial vector field λq does not appear according to $\langle\partial G/\partial v,q\rangle=0$, which is a corollary of (3.2). The function $\dot G$ is independent of the particular screen passing through $(q,\dot q)$. The equation $\dot G=0$ means that G is a first integral of (2.1), whatever be the screen $\mathcal{H}$, provided the force field f is defined on the whole semi-conic open set Ω as a positively homogeneous vector field of degree -3, in agreement with Axiom (iii). Then $\dot G$ satisfies $\dot G(q,v+\gamma q)=\dot G(q,v)$ and $\dot G(\lambda q,\lambda^{-1}v)=\lambda^{-2}\dot G(q,v)$.

If the function $G_{\mathcal{H}}(q,v)$ is positively homogeneous of degree b in v, the function $G(q,v)$ given by (3.1) is positively homogeneous of degree b in v and consequently by (3.2) positively bi-homogeneous in (q,v) with pair of degrees (b,b). The function $\dot G(q,v)$ is sum of two positively bi-homogeneous terms with respective pair of degrees $(b-1,b+1)$ and $(b-3,b-1)$.

It is a standard deduction from such an observation that if a first integral is a finite sum of terms which are positively homogeneous in the velocities, one can decompose it as a sum of first integrals of the form $G_b+G_{b-2}+\cdots+G_{b-2k}$, where the G_m are non-zero positively homogeneous functions of degree m in v, and k is a non-negative integer. The highest order term satisfies $\langle\partial G_b/\partial q,v\rangle=0$, i.e. it is a first integral of the free motion $\ddot q=\lambda q$.

3.2. *First integrals of the free motion.* Let us consider in general a first integral $R_{\mathcal{H}}$ of equation (2.1) where we set $f\equiv 0$. The extension R constructed from $R_{\mathcal{H}}$ by the process (3.1) satisfies $0=\dot R(q,v)=\langle\partial R/\partial q,v\rangle$. Consequently for any (q,v) in the domain of definition, any sufficiently small $\gamma\in\mathbb{R}$,

$$R(q,v+\gamma q)=R(q,v),\qquad R(q+\gamma v,v)=R(q,v). \tag{3.3}$$

We assume moreover than $R_{\mathcal{H}}$ is homogeneous of degree b in the velocity v. As before, for any $\lambda>0$, $R(\lambda q,v)=R(q,\lambda v)=\lambda^b R(q,v)$. If we forget the "sufficiently small" condition on γ the above relations imply

$$R(q,v)=R(q,v-q)=R(v,v-q)=R(v,-q), \tag{3.4}$$

a fundamental symmetry of exchange that can be seen only after homogenization (and that apparently was never noticed before). A way to combine this identity

with the second identity (3.2) is the statement: *a first integral of the free motion is a function of the projective impulsion* $q \wedge v$. We will re-state this precisely in the more restricted context of the polynomial first integrals.

4. First integral which are polynomial in the velocities

To study a system (2.1) one tries to obtain a complete base of first integrals. In the classical examples, it appears useless to look for first integrals which are not polynomial in v. It is not *a priori* impossible for a first integral to be rational in v, without being the quotient of two polynomial first integrals. But in most classical systems (2.1), such integral does not exist, while polynomial first integrals appear frequently, often in an unpredictable way. On the other hand, some first integrals which are real analytic in v, but not polynomial, appear for example in the completely repulsive three body problem. But they are computed through the computation of the orbits, and thus cannot give *a priori* constraints on the dynamics.

Affine case. We study a system $\ddot{q} = f(q)$, where q moves in an open set $\mathcal{H}$ of an affine space, i.e. we suppose that in (2.1) the screen function h is a linear form, and $\mathcal{H}$ is the intersection of $\Omega \subset V$ with an affine hyperplane.

4.1. *Proposition.* Let $\mathcal{H} \subset A$ be an open set in an affine space A. Let W be the tangent vector space and $f : \mathcal{H} \to W$, $q \mapsto f(q)$ a locally Lipschitzian vector field. Let $G(q, \dot{q})$ be a first integral of $\ddot{q} = f(q)$ which is a polynomial of degree b in the second variable $\dot{q}$, the coefficients being differentiable functions of q. Then the highest degree term $G_b(q, \dot{q})$ is a first integral of $\ddot{q} = 0$.

4.2. *Proposition.* Let $\mathcal{H} \subset A$ be a connected open set in an affine space A. Let W be the vector space tangent to A. Let $R_{\mathcal{H}} : \mathcal{H} \times W \to \mathrm{I\!R}$, $(q, \dot{q}) \mapsto R_{\mathcal{H}}(q, \dot{q})$ be a first integral of $\ddot{q} = 0$. Assume $R_{\mathcal{H}}(q, v)$ is a polynomial in the second variable $v \in W$ whose coefficients are arbitrary functions of $q \in \mathcal{H}$. Then $R_{\mathcal{H}}$ is a polynomial in $(q, v) \in A \times W$.

Remarks. Proposition 4.1 is standard. Proposition 4.2 improves a theorem in [Nij]. Nijenhuis assumed the continuity of $R_{\mathcal{H}}$, while we do not. A first integral is a function which is constant along any solution. This definition does not need any regularity. The proof of 4.2 is that, the first integral being polynomial in v and satisfying (3.4), it is polynomial in q, so it is a polynomial. The idea to use (3.4) to reprove and improve Nijenhuis' result comes from a discussion with Alain Chenciner. Later we will show that the projective point of view improves the description of the polynomials $R_{\mathcal{H}}$ which satisfy the hypothesis of 4.2.

Proof of 4.1. Each term of the expansion of $\langle \partial G/\partial q, v\rangle + \langle \partial G/\partial v, f\rangle$ as a polynomial in v is zero, in particular the highest degree term $\langle \partial G_b/\partial q, v\rangle$. QED

4.3. *Lemma.* Let A be an m-dimensional affine space, B an n-dimensional affine space, $\mathcal{U} \subset A$ and $\mathcal{V} \subset B$ two open sets. If $F : \mathcal{U} \times \mathcal{V} \to \mathrm{I\!R}$, $(x, y) \mapsto F(x, y)$ coincides on $\mathcal{U} \times \mathcal{V}$ with a polynomial of degree k in x whose coefficients are functions of y, and also coincides with a polynomial of degree l in y whose coefficients are functions of x, then F is the restriction to $\mathcal{U} \times \mathcal{V}$ of a polynomial in $(x, y) \in A \times B$.

Proof. Let $M = (m+k)!/m!\,k!$ and $N = (n+l)!/n!\,l!$. Let $x_1, \dots, x_M$ and $y_1, \dots, y_N$ be points in general position in $\mathcal{U}$ and $\mathcal{V}$ respectively. We mean: on A the unique polynomial function of degree at most k vanishing on the x_i's is the null polynomial. To see that there exist M points in general position in any open set of A, it is sufficient to see that there exists a set of points in general position, and that the general position property is preserved by translation and dilation.

There are polynomials $P_1, \dots, P_M$ on A such that for every polynomial P of degree at most k, and any $x \in A$, $P(x) = \sum_{i=1}^M P(x_i)P_i(x)$. The analogous polynomials for B will be called $Q_1, \dots, Q_N$. Then we have

$$F(x,y) = \sum_{j=1}^N F(x,y_j)Q_j(y) = \sum_{j=1}^N \sum_{i=1}^M F(x_i,y_j)P_i(x)Q_j(y).$$

The first equality is the hypothesis that for a given x, $F(x,y)$ is a polynomial in y of degree at most l. The second is that $F(x,y_i)$ is a polynomial in x of degree at most k. QED

Proof of 4.2. A first integral of $\ddot{q} = 0$ is a function $R_{\mathcal{H}}$ such that for any fixed $(q,v) \in \mathcal{H} \times W$, the real function $\gamma \mapsto R_{\mathcal{H}}(q + \gamma v, v)$ of the real variable γ is constant on each of its intervals of definition. Let us consider A as the affine hyperplane of a vector space V with equation $\langle h, q \rangle = 1$, where $h \in V^*$. Let $\Omega \subset V \setminus \{0\}$ be the semi-cone constructed on $\mathcal{H}$. We construct

$$\begin{aligned} R : \Omega \times V &\longrightarrow \mathbb{R} \\ (q,v) \quad &\longmapsto R_{\mathcal{H}}\Big(\frac{q}{\langle h,q\rangle}, \langle h,q\rangle v - \langle h,v\rangle q\Big). \end{aligned}$$

as in (3.1). Then R is a polynomial in v whose coefficients are functions of q. If $\langle h,q\rangle > 0$ and $\langle h,v\rangle > 0$ we set $q_0 = \langle h,q\rangle^{-1} q$, $v_0 = \langle h,v\rangle^{-1} v$, $\gamma = \langle h,q\rangle^{-1}\langle h,v\rangle^{-1}$. The expression above is $R_{\mathcal{H}}\big(q_0, \gamma^{-1}(v_0 - q_0)\big)$. Let $I_{qv} = \{\alpha q + \beta v, \alpha \geq 0, \beta \geq 0\}$. If $I_{qv} \subset \Omega$ then this expression is also $R_{\mathcal{H}}\big(v_0, \gamma^{-1}(v_0 - q_0)\big)$ because $R_{\mathcal{H}}$ is first integral of $\ddot{q} = 0$. Thus $R(v, -q) = R(q,v)$ if $I_{qv} \subset \Omega$. To guarantee this last hypothesis, we choose any semi-conic convex set $\mathcal{C} \subset \Omega$. For any $(q,v) \in \mathcal{C} \times \mathcal{C}$, we conclude that $R(q,v) = R(v,-q)$. By Lemma 4.3, if $\mathcal{C}$ is open, there is a polynomial function in (x,v) which coincides with R on $\mathcal{C} \times \mathcal{C}$. If $\mathcal{C}_1 \subset \Omega$ is another open semi-conic convex set, there exists another polynomial coinciding with R on $\mathcal{C}_1 \times \mathcal{C}_1$. If $\mathcal{C} \cap \mathcal{C}_1 \neq \{0\}$ both polynomial coincide. So by connectedness R is a polynomial function which extends to $V \times V$, and gives by restriction a polynomial on $A \times W$ coinciding with $R_{\mathcal{H}}$ on $\mathcal{H} \times W$. QED

Case of a general screen. The conclusions we just proved on the equation $\ddot{q} = f$ in the affine case pass to the equation $\ddot{q} = f + \lambda q$ on a general screen.

4.4. *Proposition.* Consider System (2.1). Let $G : T\mathcal{H} \to \mathbb{R}$ be a smooth first integral which at each $q \in \mathcal{H}$ is a polynomial on $T_q\mathcal{H}$ of degree less than $b \in \mathbb{N}$. In other words $G = G_b + G_{b-1} + \dots + G_0$, where $G_i : T\mathcal{H} \to \mathbb{R}$ is smooth and is a homogeneous polynomial of degree i when restricted to any fiber $T_q\mathcal{H}$. Then

$G = G_b + G_{b-2} + G_{b-4} + \cdots$ is a first integral of (2.1), and G_b is a first integral of $\ddot{q} = \lambda q$.

4.5. *Proposition.* Consider System (2.1) with a vector field $f \equiv 0$. Let $R_{\mathcal{H}} : T\mathcal{H} \to \mathbb{R}$ be a first integral. Assume $R_{\mathcal{H}}$ is at each $q \in \mathcal{H}$ a polynomial on $T_q\mathcal{H}$ of degree less than $b \in \mathbb{N}$. Then $R_{\mathcal{H}}$ is the restriction to $T\mathcal{H}$ of polynomial $R : V \times V \to \mathbb{R}$, $(q, \dot{q}) \mapsto R(q, \dot{q})$.

The proof is: take f and the first integral, homogenize them, restrict to an affine screen, apply 4.1 and 4.2 and go back to the initial screen. The remaining statements in 4.4 are proved in 3.1. Here we do not care about finding the least possible smoothness for the screen $\mathcal{H}$. Actually, the screen is unessential, the dynamics being defined, except for the time parameter, at the projective level.

Gerard Thompson [Thn] gave a short proof of 4.2, assuming the first integral are sufficiently differentiable.

5. General facts on tensors with Young tableau symmetry

5.1. A *Young tableau* is given by the number $r \in \mathbb{N}$ of its rows, the list $(i_1, \dots, i_r) \in \mathbb{N}^r$, $i_1 \geq i_2 \geq \cdots \geq i_r \geq 1$, of the lengths of its rows, and a way to number the boxes. For example the Young tableau $(5, 5, 3, 1)$, with four rows, "numbered horizontally", is represented here.

1	2	3	4	5
6	7	8	9	10
11	12	13		
14				

The shape of the same Young tableau may be given by the number $c = i_1$ of columns, and the list $(j_1, \dots, j_c) \in \mathbb{N}^r$, $j_1 \geq j_2 \geq \cdots \geq j_c \geq 1$, of the lengths of the columns. This is convenient when we insist on the columns. The above tableau has 5 columns with respective lengths 4,3,3,2,2.

But when we insist on the columns the simplest numbering is often the "vertical" one. So we decide that the tableau $[4, 3, 3, 2, 2]$ is *not* the above one, but:

1	5	8	11	13
2	6	9	12	14
3	7	10		
4				

The Young tableaux $(5, 5, 3, 1)$ and $[4, 3, 3, 2, 2]$ have the same shape but they are different, because the numbering is different.

5.2. Given a vector space V and a Young tableau $Y = (i_1, \dots, i_r)$, we consider the space of multilinear forms $\bigotimes^N V^*$, with $N = i_1 + \cdots + i_r$. By convention, we associate to the first variable the first box in the tableau, to the second the second box in the first row, to the $i_1 + 1$-th the first box of the second row, etc.

The order corresponds to the numbering of Y. We introduce two operators S and A, the *Young symmetrizers*, acting on $\bigotimes^N V^*$ with Y.

The operator S symmetrizes a multilinear form $\phi \in \bigotimes^N V^*$ in the variables corresponding to boxes of the first row of Y *and* to the boxes of the second row *and* to boxes of the third row, etc. For example if $Y = (2,2)$, $(S\phi)(x,y,z,t) = \phi(x,y,z,t) + \phi(y,x,z,t) + \phi(x,y,t,z) + \phi(y,x,t,z)$.

The operator A antisymmetrizes a multilinear form $\phi \in \bigotimes^N V^*$ in the boxes of the first *column* of Y and in the boxes of the second column, etc. For example if $Y = (2,2)$, $(A\phi)(x,y,z,t) = \phi(x,y,z,t) - \phi(z,y,x,t) - \phi(x,t,z,y) + \phi(z,t,x,y)$.

To write correctly such relations, we memorize that rows are associated to symmetry, the columns to antisymmetry, and we work mentally with diagrams as

$$\begin{array}{|c|c|}\hline x & y \\ \hline z & t \\ \hline\end{array} - \begin{array}{|c|c|}\hline z & y \\ \hline x & t \\ \hline\end{array} - \begin{array}{|c|c|}\hline x & t \\ \hline z & y \\ \hline\end{array} + \begin{array}{|c|c|}\hline z & t \\ \hline x & y \\ \hline\end{array}.$$

5.3. *Proposition.* For any Young tableau, there is a non-zero $\lambda \in \mathbb{N}$ such that $A : \bigotimes^N V^* \to \bigotimes^N V^*$ and $S : \bigotimes^N V^* \to \bigotimes^N V^*$ defined above satisfy $ASAS = \lambda AS$ and $SASA = \lambda SA$.

This result is due to [You], p. 364. A simplified proof by J. v. Neumann is presented in [VdW], p. 192, in [We1], p. 363 or in [We2], p. 124. These proofs are purely combinatorial and do not use any theoretical background. They concern $SASA = \lambda SA$. We obtain $ASAS = \lambda AS$ by transposition, noticing that the operator ${}^tS$ acts on $\bigotimes^N V$ as S acts on $\bigotimes^N V^*$, i.e. by the same process of symmetrization, and that the same is true for A.

5.4. *Proposition.* Let $Y = [j_1, j_2, \ldots, j_c]$ be a Young tableau with c columns, the boxes being numbered vertically. Let $N = j_1 + \cdots + j_c$. The image of AS is the subspace of $\bigotimes^N V^*$ generated by the elements $(\xi_1 \wedge \cdots \wedge \xi_{j_1}) \otimes (\xi_1 \wedge \cdots \wedge \xi_{j_2}) \otimes \cdots \otimes (\xi_1 \wedge \cdots \wedge \xi_{j_c})$, where $(\xi_1, \ldots, \xi_{j_1}) \in (V^*)^{j_1}$. A $\phi \in \mathrm{Im}AS$ is zero if for any $(x_1, \ldots, x_{j_1}) \in V^{j_1}$,

$$\phi(x_1, x_2, \ldots, x_{j_1}; x_1, x_2, \ldots, x_{j_2}; \ldots; x_1, x_2, \ldots, x_{j_c}) = 0. \tag{5.1}$$

Proof. The image of S is generated by the elements $\xi_1 \otimes \cdots \otimes \xi_{j_1} \otimes \xi_1 \otimes \cdots \otimes \xi_{j_2} \otimes \cdots \otimes \xi_1 \otimes \cdots \otimes \xi_{j_c}$, i.e. elements having the same covector repeated along a same row. This proves the first claim. By the same argument (5.1) means that $\langle \phi, x \rangle = 0$ for an $x \in \bigotimes^N V$ obtained by Young symmetrization from any element of $\bigotimes^N V$. This means $S\phi = 0$. But there exists ψ such that $\phi = AS\psi$. So $ASAS\psi = 0 = \lambda AS\psi = \lambda\phi = 0$. QED

5.5. Taking again the example $Y = (2,2)$, numbered horizontally, the image of AS contains quadrilinear forms that are antisymmetric in the first and third boxes on one side, in the second and the fourth boxes on the other side. Furthermore, as they are obtained after antisymmetrization $(1,3)$ of a tensor that was symmetric in $(1,2)$, the antisymmetric $(1,2,3)$ part is zero. *The tensor satisfies the so-called algebraic Bianchi identity.* We arrived at the famous space

of quadrilinear forms where the Riemann curvature tensor lives.

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}$$

The operator S establishes an isomorphism between the image of AS and the image of SA. In the example above, S transforms the Riemann tensor in the *symmetrized Riemann tensor* (see e.g. [Syn], p. 54). It is a tensor that contains exactly the information of the Riemann tensor, but is symmetric in $(1,2)$, in $(3,4)$, and returns zero when symmetrized in $(1,2,3)$.

The above algebraic Bianchi identity, together with the antisymmetries, characterizes the image of AS. This is a particular case of a statement true for any Young tableau. In Proposition 5.7 we give this general statement. In Proposition 5.6 we give the analogous statement about the image of SA. This statements are useful in many applications, and are often proved in particular cases. The general statement appears, in a slightly weaker version than ours, at the bottom of p. 145 of [PeR]. This was indicated to me by José María Pozo Soler, who also helped me to prepare this section as well as Sections 8 and 9. As we were not able to find the proofs in any place, and used the results in several particular cases, we decided to write them.

5.6. *Proposition.* Let $Y = (i_1, \ldots, i_r)$ be a Young tableau with r rows, the boxes being numbered horizontally. Let $N = i_1 + \ldots + i_r$. Let $(s_1, \ldots, s_r)$ be the numbers in the boxes of the first column: $s_k = i_1 + \cdots + i_{k-1} + 1$. For any (m,n), $1 \le m < n \le N$, we denote by $\mathcal{T}_m^n$ the transposition operator acting on $\bigotimes^N V^*$, exchanging the variables m and n of an N-linear form $\phi \in \bigotimes^N V^*$. An N-linear form $\phi \in \bigotimes^N V^*$ is element of $\operatorname{Im} SA$ if and only if

(i) it is symmetric in each row: for any row k, for any (m,n), $s_k \le m < n < s_{k+1}$, $\mathcal{T}_m^n \phi = \phi$,

(ii) it satisfies $r-1$ "algebraic Bianchi identities": for any row k, $1 \le k < r$,

$$\phi + \mathcal{T}_{s_k}^{s_{k+1}} \phi + \mathcal{T}_{s_k+1}^{s_{k+1}} \phi + \cdots + \mathcal{T}_{s_k+i_k-1}^{s_{k+1}} \phi = 0.$$

5.7. *Proposition.* Let $Y = [j_1, \ldots, j_c]$ be a Young tableau with c columns, the boxes being numbered vertically. Let $N = j_1 + \ldots + j_c$. Let $(s_1, \ldots, s_c)$ be the numbers in the boxes of the first row: $s_k = j_1 + \cdots + j_{k-1} + 1$. An N-linear form $\phi \in \bigotimes^N V^*$ is element of $\operatorname{Im} AS$ if and only if

(i) it is antisymmetric in each column: for any column k, for any (m,n), $s_k \le m < n < s_{k+1}$, $\mathcal{T}_m^n \phi = -\phi$,

(ii) it satisfies $r-1$ "algebraic Bianchi identities": for any column k, $1 \le k < c$,

$$\phi - \mathcal{T}_{s_k}^{s_{k+1}} \phi - \mathcal{T}_{s_k+1}^{s_{k+1}} \phi - \cdots - \mathcal{T}_{s_k+i_k-1}^{s_{k+1}} \phi = 0.$$

Remarks. The way (ii) to write the symmetrization or antisymmetrization is unusual but quite efficient. In each statement, the "only if" part is easy, not

more difficult than in the case $Y = (2,2)$ we just treated. We now prove the "if" part. The arguments look quite different in both proofs.

Proof of 5.6. Consider a ϕ satisfying the hypothesis and compute, for arbitrary $(x_1, \dots x_r) \in V^r$,

$$(A\phi)(x_1, \dots, x_1; x_2, \dots, x_2; \dots; x_r, \dots, x_r). \tag{5.2}$$

We mean we "fill up" $A\phi$, in the example $Y = (5,5,3,1)$, as follows

$$\begin{array}{ccccc} x_1 & x_1 & x_1 & x_1 & x_1 \\ x_2 & x_2 & x_2 & x_2 & x_2 \\ x_3 & x_3 & x_3 & & \\ x_4 & & & & \end{array}$$

The value of $A\phi$ is a sum with signs of values of ϕ. In each term, the positions of the variables on each column are exchanged. We consider any of these terms, e.g. the one corresponding to

$$\begin{array}{ccccc} x_3 & x_1 & x_3 & x_1 & x_2 \\ x_2 & x_3 & x_1 & x_2 & x_1 \\ x_4 & x_2 & x_2 & & \\ x_1 & & & & \end{array}$$

If we apply the last of the identities (ii), this term is changed into three terms, none of which possess x_1 in the bottom row. Such operation would work also if there were already some x_1 in the third row. In this case several terms in the Bianchi identity (ii) would be equal. We would simply put them all in the same side of the equation, to get their common value as a function as the sum of the other terms, divided by an integer. We claim that repeating this operation, we create new terms where the x_1's are higher and higher, to finish with terms where they are all "at their place", in the first row. When necessary we put x_1 in the first column using a transposition (i). We then place the x_2's in the second row, etc. Finally we express (5.2) as a sum where each term is a rational number times $\phi(x_1, \dots, x_1; \dots; x_r, \dots, x_r)$. There is a rational number μ which does not depend neither on ϕ nor on the x_i's, such that $(A\phi - \mu\phi)(x_1, \dots, x_1; \dots; x_r, \dots, x_r) = 0$. This is true for any $(x_1, \dots, x_r) \in V^r$ so $S(A\phi - \mu\phi) = 0$. As ϕ satisfies (i), $S\phi = i_1!i_2! \dots i_r!\phi$. Thus ϕ is a multiple of $SA\phi$ and $\phi \in \mathrm{Im}SA$. It remains to check that $\mu \neq 0$. But clearly $i_1!i_2! \dots i_r!\mu = \lambda$ according to Proposition 5.3. QED

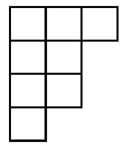
5.8. *Lemma.* Under the hypothesis of Proposition 5.7, we have

(iii) ϕ satisfies $r(r-1)/2$ algebraic Bianchi identities: for any (j,k), $1 \le k < j \le c$,

$$\phi - \mathcal{T}_{s_k}^{s_j}\phi - \mathcal{T}_{s_k+1}^{s_j}\phi - \dots - \mathcal{T}_{s_k+i_k-1}^{s_j}\phi = 0.$$

Proof. This states the "transitivity" of algebraic Bianchi identities. It is sufficient to prove the case of three columns, $c = 3$. As only the upper box of the

third column is involved, we can suppose $j_3 = 1$.



According to (ii) between Columns 2 and 3, and (i) on Column 2, we write

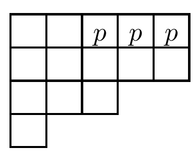
$$\begin{aligned}&\phi(x_1, \dots; y_1, \dots, y_{j_2}; z) = \\ &\quad = -\phi(x_1, \dots; z, y_2, \dots, y_{j_2}; y_1) + \phi(x_1, \dots; z, y_1, y_3, \dots, y_{j_2}; y_2) - \cdots\end{aligned}$$

Then we expand each term in the same way using the other identity (ii)

$$\begin{aligned}&-\phi(x_1, \dots; z, y_2, \dots, y_{j_2}; y_1) = \\ &\quad = \phi(z, x_2, \dots; x_1, y_2, \dots; y_1) - \phi(z, x_1, x_3, \dots; x_2, y_2, \dots; y_1) + \cdots \\ &\phi(x_1, \dots; z, y_1, y_3, \dots, y_{j_2}; y_2) = \\ &\quad = -\phi(z, x_2, \dots; x_1, y_1, \dots; y_2) + \phi(z, x_1, x_3, \dots; x_2, y_1, \dots; y_2) - \cdots\end{aligned}$$

The first terms of the right hand sides give $-\phi(z, x_2, \dots; y_1, y_2, \dots; x_1)$, the second terms together give $\phi(z, x_1, \dots; y_1, y_2, \dots; x_2)$, etc. All together gives the Bianchi identity between Columns 1 and 3. QED

5.9. *Lemma.* Let $Y = [j_1, \dots, j_c]$, $N = j_1 + \dots + j_c$ and $\phi \in \otimes^N V^*$ satisfying (i) and (ii) of Proposition 5.7. Let $p \in V$ be an arbitrary vector. Then for any integer l, $1 \le l \le c$, the $N - l$-linear form ϕ' obtained by contracting p at the top of the last l columns satisfies Relations (i) and (ii) of Proposition 5.7 for the Young tableau $Y' = [j_1, \dots, j_{c-l}, j_{c-l+1} - 1, \dots, j_c - 1]$.



Proof. Obviously ϕ' satisfies Relation (i). Relation (ii) if $k < c - l$ is the same equation for ϕ and ϕ'. Suppose $k = c - l$. If $j_{k+1} = 1$, the column k in Y' does not exist and no Relation (ii) is needed. If $j_{k+1} \ge 2$, Relation (ii) for Y can be written using the second top box in Column $k + 1$ instead of the first. It is the same equation for ϕ and ϕ'. If $k > c - l$, there is one term less for ϕ', but this term is zero, because p is repeated in the column $k + 1$. QED

Proof of 5.7. Let $\mathcal{A}$ be the subspace of $\bigotimes^N V^*$ defined by (i) and (ii). Note that (i) means $\mathcal{A} \subset \mathrm{Im}A$. We know that $\mathrm{Im}AS \subset \mathcal{A}$ and that S establishes a bijection between $\mathrm{Im}AS$ and $\mathrm{Im}SA$. It is thus sufficient to prove that $S : \mathcal{A} \to \mathrm{Im}SA$ is injective. So we want to prove that if $\phi \in \mathcal{A}$ is such that $S\phi = 0$, then $\phi = 0$. The hypothesis $S\phi = 0$ is equivalent to:

$$\phi(q_1, \dots, q_{j_1}; q_1, \dots, q_{j_2}; \dots; q_1, \dots, q_{j_c}) = 0 \quad \text{for any } (q_1, \dots, q_{j_1}) \in V^{j_1}. \tag{5.3}$$

We want to prove: (5.3) $\Rightarrow \phi = 0$. We first prove by induction on n that for any n, $1 \leq n \leq j_1$,

$$X_n = \phi(q_1, \dots, q_{j_1}; \dots; q_1, \dots, q_{j_{k-1}}; q_n, q_2, \dots, q_{j_k}) = 0. \tag{5.4}$$

We know that $X_1 = 0$. We write, for each $i < n$, the derivative of X_i with respect to the variable q_i, in the direction of q_n. This derivative vanishes by the induction hypothesis $X_i = 0$. We get the system:

$$\begin{aligned} 0 = \partial_1 X_1 \cdot q_n &= Y_1^1 + \cdots + Y_1^{k-1} + X_n, \\ 0 = \partial_2 X_2 \cdot q_n &= Y_2^1 + \cdots + Y_2^{k-1} + X_n, \\ &\vdots \\ 0 = \partial_{n-1} X_{n-1} \cdot q_n &= Y_{n-1}^1 + \cdots + Y_{n-1}^{k-1} + X_n, \end{aligned} \tag{5.5}$$

where $Y_i^l = \phi(\dots; q_1, \dots, q_{j_{l-1}}; q_1, \dots, q_{i-1}, q_n, q_{i+1}, \dots, q_{j_l}; \dots; q_i, q_2, \dots, q_{j_k})$. If $i > j_l$ then $Y_i^l = 0$. The Bianchi identity (iii) between the columns k and l is $X_n = Y_1^l + \cdots + Y_{j_l}^l$. So the sum $Y_1^l + \cdots + Y_{n-1}^l$ is zero if $n \leq j_l$, because it has only vanishing terms, and it is X_n if $n > j_l$. When we add all the equations (5.5), column by column, we find X_n times a positive integer. Thus (5.4) holds.

The previous argument may be used once more. We write the system $\partial_1 X_1 \cdot p = \partial_2 X_2 \cdot p = \cdots = \partial_{j_1} X_{j_1} \cdot p = 0$, sum all the terms and use the Bianchi identity. We obtain

$$X = \phi(q_1, \dots, q_{j_1}; \dots; q_1, \dots, q_{j_{k-1}}; p, q_2, \dots, q_{j_k}) = 0, \quad \text{for any } p \in V. \tag{5.6}$$

We can illustrate what we just proved by the following diagram.

$$0 = \begin{array}{|c|c|c|c|c|} \hline q_1 & q_1 & q_1 & q_1 & q_1 \\ \hline q_2 & q_2 & q_2 & q_2 & q_2 \\ \hline q_3 & q_3 & q_3 & \multicolumn{2}{c}{} \\ \cline{1-3} q_4 & \multicolumn{4}{c}{} \\ \cline{1-1} \end{array} \qquad \Longrightarrow \qquad 0 = \begin{array}{|c|c|c|c|c|} \hline q_1 & q_1 & q_1 & q_1 & p \\ \hline q_2 & q_2 & q_2 & q_2 & q_2 \\ \hline q_3 & q_3 & q_3 & \multicolumn{2}{c}{} \\ \cline{1-3} q_4 & \multicolumn{4}{c}{} \\ \cline{1-1} \end{array}$$

We observe now that (5.6) is the equation (5.3) written for the $N-1$-linear form ϕ' obtained contracting $p \in V$ in ϕ, at the top of the last column. This form satisfies (i) and (ii) by Lemma 5.9. We prove our assertion (5.3) $\Rightarrow \phi = 0$ by induction on the number of boxes N of the Young tableau. QED

5.10. *Example.* Consider the space $\mathcal{B} \subset \bigotimes^4 V^*$ of the quadrilinear forms ϕ such that, for any $(x,y,z,t) \in V^4$, $\phi(x,y,z,t) = \phi(z,t,x,y)$, $\phi(x,y,z,t) = -\phi(y,x,z,t)$ and $\phi(x,y,z,t) = -\phi(x,y,t,z)$. Suppose $\phi \in \mathcal{B}$ and $\phi(x,y,x,y) = 0$ for any $(x,y) \in V^2$. Then $\phi \in \bigwedge^4 V^*$, i.e. ϕ is completely antisymmetric.

Proof. Consider the projector $B : \mathcal{B} \to \mathcal{B}$, $\phi \mapsto \psi$ where ψ is defined by $\psi(x,y,z,t) = \phi(x,y,z,t) + \phi(y,z,x,t) + \phi(z,x,y,t)$. As $B^2 = 3B$, $\mathcal{B} = \ker B \oplus \mathrm{Im} B$. We check directly that ψ is antisymmetric so $\mathrm{Im} B = \bigwedge^4 V^*$. By Proposition 5.7, $\ker B = \mathrm{Im} AS$ for the Young tableau $Y = [2,2]$ numbered vertically. We decompose $\phi = \phi_Y + \psi/3$ according to the decomposition of $\mathcal{B}$. Finally we write $\phi(x,y,x,y) = \phi_Y(x,y,x,y) = 0$. Thus $\phi_Y = 0$ by 5.4. QED

6. The Young tableau symmetry of the highest degree term

The highest degree term of a first integral which is a polynomial in the velocities with coefficients depending on the position, is extended by the homogenization process in a polynomial $(q, \dot{q}) \mapsto R(q, \dot{q})$ with the fundamental properties: (i) for any $\lambda \in \mathbb{R}$, $R(\lambda q, \dot{q}) = R(q, \lambda \dot{q}) = \lambda^b R(q, \dot{q})$, where $b \in \mathbb{N}$ is the degree, and (ii) for any $\gamma \in \mathbb{R}$, $R(q, \dot{q} + \gamma q) = R(q, \dot{q})$, (iii) $R(q + \gamma \dot{q}, \dot{q}) = R(q, \dot{q})$. The last property will appear to be a consequence of the first two.

6.1. *Definition.* For any $b \in \mathbb{N}$, we call $\mathcal{P}^{b,b}(V)$ the space of polynomials $R : V \times V \to \mathbb{R}$, $(q, v) \mapsto R(q, v)$ which (i) are homogeneous of degree b in each variable and (ii) satisfy $R(q, v + \gamma q) = R(q, v)$ for any $(q, v, \gamma) \in V \times V \times \mathbb{R}$.

We translate these properties into properties of the "polar form" $R_{\mathcal{S}}$ of R. It is the $2b$-linear form on V, symmetric in the first b arguments, symmetric in the last b arguments, such that:

$$R(q, v) = R_{\mathcal{S}}(q, \ldots, q; v, \ldots, v). \tag{6.1}$$

The polar form $R_{\mathcal{S}}$ is obtained, as well-known, by a repeated differentiation of the polynomial R. Condition (i) fixes the number $2b$ of arguments. Condition (ii) may be written $dR(q, v + \gamma q)/d\gamma = 0$, which gives

$$R_{\mathcal{S}}(q, \ldots, q; q, v, \ldots, v) = 0, \qquad \text{for any } (q, v) \in V \times V. \tag{6.2}$$

6.2. *Proposition.* Let $b \in \mathbb{N}$ and $R_{\mathcal{S}}$ be a $2b$-linear form, symmetric in the first b variables, symmetric in the last b variables. The polynomial R given by expression (6.1) is in $\mathcal{P}^{b,b}(V)$ if and only if (6.2) is satisfied.

The proof is what we just said. Condition (6.2) is equivalent to: the symmetrization of $R_{\mathcal{S}}$ in its $b+1$ first arguments gives zero. By Proposition 5.6, it implies $R_{\mathcal{S}}$ has a symmetry with Young tableau (b, b). We can state this as follows.

6.3. *Proposition.* A polynomial R is in $\mathcal{P}^{b,b}(V)$ if and only if there exists a $2b$-linear form $R_{\mathcal{A}}$ which is, for any i, $1 \leq i \leq b$, antisymmetric by exchange of the $2i-1$-th and $2i$-th variable, such that

$$R(q, v) = R_{\mathcal{A}}(q, v; q, v; \ldots; q, v). \tag{6.3}$$

Consequently an $R \in \mathcal{P}^{b,b}(V)$ satisfies $R(q, v) = (-1)^b R(v, q)$.

q	q	q	q	q
v	v	v	v	v

6.4. *Definition.* We call $\mathcal{P}^{b,b}_{\mathcal{S}}(V)$ the space of $2b$-linear $R_{\mathcal{S}}$ forms satisfying the symmetry condition of Proposition 6.2 and the condition (6.2). We call $\mathcal{P}^{b,b}_{\mathcal{A}}(V)$ the space of $2b$-linear forms $R_{\mathcal{A}}$ satisfying the antisymmetry condition of Proposition 6.3, and the

6.5. *Algebraic Bianchi identities.* For any i, $1 \leq i \leq b-1$, the antisymmetrization of $R_{\mathcal{A}}$ in the $2i-1$-th, $2i$-th and $2i+1$-th variables gives zero.

So, $\mathcal{P}^{b,b}_{\mathcal{S}}(V) = \mathrm{Im} SA$, S and A being the Young symmetrizers associated to the Young tableau (b, b) numbered horizontally. And according to Proposition 5.7, $\mathcal{P}^{b,b}_{\mathcal{A}}(V) = \mathrm{Im} AS$, where A and S are the Young symmetrizers associated to the Young tableau with same shape but numbered vertically. This may be stated as follows.

6.6. *Proposition.* If $R \in \mathcal{P}^{b,b}(V)$, there exists a unique $R_{\mathcal{A}} \in \mathcal{P}^{b,b}_{\mathcal{A}}(V)$ satisfying (6.3). The $2b$-linear form $R_{\mathcal{A}}$ defines a unique b-linear form on $\bigwedge^2 V$, which is symmetric. It defines uniquely a polynomial $R_{\mathcal{B}} : \bigwedge^2 V \to \mathbb{R}$, homogeneous of degree b, such that $R_{\mathcal{B}}(q \wedge v) = R(q, v)$. The polynomial $R_{\mathcal{B}}$ is completely determined by its values on the decomposable bivectors.

Proof. By 5.4, $R_{\mathcal{A}}$ is linear combination of terms $(\xi_1 \wedge \xi_2) \otimes (\xi_1 \wedge \xi_2) \otimes \cdots \otimes (\xi_1 \wedge \xi_2)$, with $(\xi_1, \xi_2) \in (V^*)^2$. So it is linear combination of terms $\omega \otimes \cdots \otimes \omega$ with $\omega \in \bigwedge^2 V^* = (\bigwedge^2 V)^*$. So it is also the symmetric b-linear form announced. The second part of 5.4 gives the last statement. QED

Clearly $\mathbb{R} \oplus \mathcal{P}^{1,1}(V) \oplus \mathcal{P}^{2,2}(V) \oplus \cdots$ is an algebra for the multiplication of polynomials. The dimension of $\mathcal{P}^{b,b}(V)$ is $n(n+1)^2(n+2)^2 \cdots (n+b-1)^2(n+b)/b!(b+1)!$ if $\dim V = n+1$. This is a result explained in [PeR]. It was proved in 1954 in [FRT]. As a result about the dimension of the space of the first integrals of the free motion, it was stated and proved, without mention to a Young tableau symmetry, in [KaL]. The result in the interesting particular case $b = 2$ was derived earlier in [Ths].

Through $R_{\mathcal{B}}$ the polynomial R may be interpreted as a polynomial on the Grassmannian of vectorial 2-planes in V. This interpretation generalizes to p-planes, $p \geq 2$ (see [FuH]). More significant to us, R appears as the most general polynomial in the projective impulsion $q \wedge v$. We have established that a polynomial first integral of the free motion, i.e. the motion with constant projective impulsion, is simply a “polynomial in the projective impulsion”.

7. Which quadratic first integrals are the Hamiltonian for a screen?

7.1. An *adapted local chart* of the tangent bundle TM is obtained by differentiation from a local chart

$$\Phi : \Omega \longrightarrow \mathbb{R}^n, \qquad q \longmapsto \Phi(q) = (x_1, \ldots, x_n),$$

of the n-dimensional manifold M, where $\Omega \subset M$ is an open set. The adapted local chart is

$$\Phi_* : T\Omega \longrightarrow \mathbb{R}^{2n}, \qquad \xi \longmapsto \big(\Phi(q), d\Phi|_q(\xi)\big) = (x_1, \ldots, x_n, y_1, \ldots, y_n),$$

where $T\Omega \subset TM$ is the inverse image of Ω by the canonical projection $TM \to M$, and $q \in \Omega$ is the image of $\xi \in T\Omega$ by this projection.

7.2. A *second order differential equation* on a manifold M is a vector field Z on TM such that $\dot{x}_i = y_i$ for any i, $1 \leq i \leq n$, and for any adapted chart. By $\dot{x}$ we mean the derivative of the coordinate function x_i along the vector field Z.

7.3. A *pre-Lagrangian* for the second order differential equation Z is a function $L : TM \to \mathbb{R}$ such that in any adapted chart $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ Lagrange equations

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial y_i}\Big) - \frac{\partial L}{\partial x_i} = 0$$

are satisfied. Here d/dt is the derivation along the vector field Z.

We do not require that the Lagrange equations define the system Z. We just require they are satisfied. The pre-Lagrangians form a vector space, which includes the functions $\sum_{i=1}^{n} \eta_i y_i$, where the η_i's are the coordinates of a closed 1-form on M. An example of second order differential equation with many quadratic pre-Lagrangians is the harmonic oscillator on $\mathbb{R}^n$, with equations $\ddot{x}_i = -x_i$, $1 \le i \le n$. The functions $\dot{x}_i\dot{x}_j - x_i x_j$, $1 \le i \le j \le n$, are pre-Lagrangians.

The following proposition is well-known in the case of Lagrangians, and the proof is not more complicated in the case of pre-Lagrangians.

7.4. *Proposition.* Let L be a pre-Lagrangian for a second order differential equation Z. Given an adapted chart $(x_1, \ldots, x_n, y_1, \ldots, y_n)$, let $p_i = \partial L/\partial y_i$. The function $\sum_{i=1}^{n} p_i y_i$ does not depend on the chart and $E = \sum_1^n p_i y_i - L$ is a first integral of Z.

7.5. *Proposition.* Let Z be a second order differential equation and $G : TM \to \mathbb{R}$ be a function. Using any local chart $(x_1, \ldots, x_n)$ of M, we define $G_* : TM \to T^*M$ by its local expression $(x_1, \ldots, x_n, y_1, \ldots, y_n) \mapsto (x_1, \ldots, x_n, p_1, \ldots, p_n)$, where $p_i = \partial G/\partial y_i$. Let ω be the pre-symplectic form on TM, pull-back by G_* of the canonical symplectic form on T^*M. Let $\Omega \subset M$ be a simply connected open set and $T\Omega \subset TM$ its inverse image by the canonical projection $\pi : TM \to M$. The Lie derivative $\mathcal{L}_Z\omega$ vanishes on $T\Omega$ if and only if there exists a $U : \Omega \to \mathbb{R}$ such that $L = G + U \circ \pi$ is a pre-Lagrangian of Z on $T\Omega$.

Remark. If M is not simply connected and $\mathcal{L}_Z\omega = 0$ on TM, then there exists a multivalued pre-Lagrangian for Z. It has the form $G + U \circ \pi$, where U is a multivalued function on M.

Proof. We write

$$p_i = \frac{\partial G}{\partial y_i}, \quad \omega = \sum_i dp_i \wedge dx_i, \quad Z\rfloor\omega = \sum_i (\dot{p}_i dx_i - y_i dp_i),$$

where

$$\sum_i y_i dp_i = d\Big(\sum_i y_i p_i\Big) - \sum_i p_i dy_i = d\Big(\sum_i y_i p_i - G\Big) + \sum_i \frac{\partial G}{\partial x_i} dx_i.$$

Erasing the exact form, and using $\mathcal{L}_Z\omega = d(Z\rfloor\omega)$ we get: ω is preserved if and only if

$$\sigma = \sum_i \big(\dot{p}_i - \frac{\partial G}{\partial x_i}\big) dx_i$$

is closed. In particular, the coefficients must be independent of the y_i's. Locally there exists a function $U(q)$ such that

$$\dot{p}_i = \frac{\partial(G+U)}{\partial x_i}.$$

7.6. *More specific systems.* We continue denoting by $\xi \in TM$ the state of the particle and $q \in M$ its position, i.e. the canonical projection of ξ. Among the second order equations on M, those of the form

$$D_\xi \xi = f(q), \tag{7.1}$$

where D is a torsion-free linear connection, and f is a tangent vector field on M, have simple properties. Here ξ denotes also the velocity field along the trajectory. Let L be a pre-Lagrangian for this equation, which is polynomial in the velocity. Then both the even and the odd part of L are polynomial pre-Lagrangians. If L is even and of degree two, then $L = T + U$, where T is a quadratic form in the velocity ξ, and U only depends on the position q. There exists a bilinear symmetric form g on M such that $2T(\xi) = g(\xi, \xi)$. Lagrange's equations are equivalent to: $Dg = 0$ and $g(f, .) = dU$.

7.7. *Yet more specific systems.* We pass to screen dynamics. System (2.1) on the screen $\mathcal{H}$ is of type (7.1). The connection on $\mathcal{H}$ is induced by the affine connection on ξ and the splitting $V = T_q\mathcal{H} \oplus [q]$ at any $q \in \mathcal{H}$. By $[q] \subset V$ we mean the vector line generated by q. If D is this linear torsion-free connection, System (2.1) is $D_\xi \xi = f(q)$, where ξ is the velocity field along the trajectory. The above conclusion applies.

7.8. *Proposition.* If System (2.1) on the screen $\mathcal{H}$ possesses the quadratic pre-Lagrangian $L = T + U$, the free motion $\ddot{q} = \lambda q$ on $\mathcal{H}$ possesses the quadratic pre-Lagrangian T. As a first integral of the free motion, T is homogenized by Formula (3.1) in an $R \in \mathcal{P}^{2,2}(V)$.

Which are the elements $R \in \mathcal{P}^{2,2}(V)$ and the screens $\mathcal{H}$ such that the restriction of R to $\mathcal{H}$ is a pre-Lagrangian for the free motion? Such an R and such an $\mathcal{H}$ are said to be *compatible*. We will establish some algebraic propositions in the next two sections before attacking this question.

8. Maps from decomposable bivectors to decomposable bivectors

Here we prove a result of linear algebra. The statement is more than sufficient to establish the result we need in Section 9. We recall that a bivector $\pi \in \bigwedge^2 V$ is called decomposable if there exists $(x, y) \in V^2$ such that $\pi = x \wedge y$. We recall that: "π is decomposable" $\Leftrightarrow \pi \wedge \pi = 0 \Leftrightarrow \operatorname{rk} \pi \leq 2$.

8.1. *Theorem.* Let V and W be to real vector spaces with same finite dimension. Let $\mathcal{R} : \bigwedge^2 V \to \bigwedge^2 W$ be a linear map that sends decomposable bivectors on decomposable bivectors. Then either (i) there exists a non-zero $\phi \in W$ such that $\mathcal{R}(\pi) \wedge \phi = 0$ for any $\pi \in \bigwedge^2 V$ or (ii) there exists a non-zero $\zeta \in W^*$ such that $\zeta \lrcorner \mathcal{R}(\pi) = 0$ for any $\pi \in \bigwedge^2 V$ or (iii) there exists an invertible linear map

$B : V \to W$ and $\varepsilon = \pm 1$ such that $\mathcal{R}(x \wedge y) = \varepsilon B(x) \wedge B(y)$ for any $(x, y) \in V^2$ or (iv) $\dim V = \dim W = 4$ and there is a non-zero $\mu \in \bigwedge^4 W$ and an invertible linear map $C : V \to W^*$ such that $\mathcal{R}(x \wedge y) = \big(C(x) \wedge C(y)\big) \lrcorner \mu$.

Remark. Conditions (iii) and (iv) are sufficient and mutually exclusive. They exhaust the case of an invertible $\mathcal{R}$. Condition (i) is sufficient but (ii) is not and suggests the study of the same problem with $\dim W < \dim V$. This section is dedicated to the proof of 8.1.

We set $\dim V = \dim W = n + 1$ and study the linear maps $\mathcal{R} : \bigwedge^2 V \to \bigwedge^2 W$ that send decomposable bivectors on decomposable bivectors, i.e.

$$\text{for any } (x, y) \in V^2, \quad \mathcal{R}(x \wedge y) \wedge \mathcal{R}(x \wedge y) = 0. \tag{8.1}$$

The simplest way to obtain such an $\mathcal{R}$ is to take a linear application $B : V \to W$, and form the unique operator $B^{\wedge 2} : \bigwedge^2 V \to \bigwedge^2 W$ characterized by

$$B^{\wedge 2}(x \wedge y) = B(x) \wedge B(y). \tag{8.2}$$

Not all the operators satisfying (8.1) are obtained in this way. For example, if $n = 2$, i.e. $\dim V = 3$, (8.1) is satisfied, but if $\operatorname{rk} \mathcal{R} = 2$, $\mathcal{R}$ cannot be a $B^{\wedge 2}$.

8.2. *Lemma.* Let $S : \bigwedge^2 V \times \cdots \times \bigwedge^2 V \to \mathbb{R}$, $(\pi_1, \ldots, \pi_p) \mapsto S(\pi_1, \ldots, \pi_p)$ be a p-linear symmetric form. The following properties are equivalent

(i) $S(\sigma, \ldots, \sigma) = 0$ for any bivector σ with $\operatorname{rk} \sigma \le 2p - 2$,

(ii) $S(\pi_1, \ldots, \pi_{p-2}, \pi, \pi) = 0$ for any decomposable bivectors $\pi_1, \ldots, \pi_{p-2}$, π,

(iii) there exists a $T \in \bigwedge^{2p} V^*$ such that $S(\pi, \ldots, \pi) = T(\pi \wedge \cdots \wedge \pi)$.

Proof. Under condition (i) $S(\sigma, \ldots, \sigma) = 0$ if $\sigma = x_1\pi_1 + \cdots + x_{p-2}\pi_{p-2} + x\pi$, where $\pi_1, \ldots, \pi_{p-2}$ and π are decomposable and $(x_1, \ldots, x_{p-2}, x) \in \mathbb{R}^{p-1}$. As all the coefficients, the coefficient of $x_1 \cdots x_{p-2} x^2$ in the expansion of $S(\sigma, \ldots, \sigma)$ is zero. It is a positive integer times $S(\pi_1, \ldots, \pi_{p-2}, \pi, \pi)$. This proves (i) $\Rightarrow$ (ii). All the coefficients in this expansion have a repeated bivector. This proves (ii) $\Rightarrow$ (i). We consider the $2p$-linear form t defined by $t(u_1, \ldots, u_{2p}) = S(u_1 \wedge u_2, \ldots, u_{2p-1} \wedge u_{2p})$. Assuming (ii), $t(u_1, u_2, u_1, u_2, u_3, u_4, \ldots, u_{2p-2}) = 0$. By 5.10, t is antisymmetric in its first four arguments. In the same way t is antisymmetric in other similar sets of four arguments, and thus completely antisymmetric. This proves (ii) $\Rightarrow$ (iii). The remaining implication is standard.

8.3. *Lemma.* An $\mathcal{R}$ satisfying (8.1) defines for any $p \ge 2$ a linear application $\mathcal{R}^{(p)} : \bigwedge^{2p} V \to \bigwedge^{2p} W$ satisfying $\mathcal{R}^{(p)}(\pi_1 \wedge \cdots \wedge \pi_p) = \mathcal{R}(\pi_1) \wedge \cdots \wedge \mathcal{R}(\pi_p)$.

Proof. Use (ii) $\Rightarrow$ (iii) in Lemma 8.2, with a $\bigwedge^{2p} W$-valued instead of a real valued form. QED

As a result, we can substitute Condition (8.1) by:

$$\begin{aligned} &\text{there exists a linear application } \mathcal{R}^{(2)} : \bigwedge^4 V \to \bigwedge^4 W \text{ such that,} \\ &\text{for any } (\pi, \sigma) \in \bigwedge^2 V \times \bigwedge^2 V, \quad \mathcal{R}(\pi) \wedge \mathcal{R}(\sigma) = \mathcal{R}^{(2)}(\pi \wedge \sigma). \end{aligned} \tag{8.3}$$

This new formulation is more convenient. It gives immediately the result:

8.4. *Proposition.* If $\mathcal{R}$ satisfying (8.1) is invertible, it maps bijectively the decomposable bivectors of V on the decomposable bivectors of W.

Proof. As $\mathcal{R}$ is surjective, $\mathcal{R}^{(2)}$ reaches all the vectors of a standard basis of $\bigwedge^4 W$, i.e. $\mathcal{R}^{(2)}$ is surjective, and thus invertible. Clearly $\left(\mathcal{R}^{(2)}\right)^{-1} = (\mathcal{R}^{-1})^{(2)}$, i.e. $\mathcal{R}^{-1}$ satisfies (8.3) and thus (8.1). QED

8.5. *The bijective case.* If $\mathcal{R}$ is bijective and $n \geq 4$, $\mathcal{R}$ defines a bijection $\mathcal{R}_{\mathcal{P}} : \mathcal{P}(V) \to \mathcal{P}(W)$, as follows. If $v \in V$ is a non-zero vector, it defines a line $[v] \subset V$ and an n-dimensional subspace $[v] \wedge V \subset \bigwedge^2 V$. We claim that the image by $\mathcal{R}$ of such a subspace is a $[w] \wedge W$ for some non-zero $w \in W$. This fact follows from the:

8.6. *Lemma.* Let V be a vector space, $\dim V = n+1$. Let $H \subset \bigwedge^2 V$ be a vector subspace such that (i) any $\pi \in H$ is decomposable, (ii) if a subspace $K \subset \bigwedge^2 V$ contains H and has the same property, then $H = K$. Either there exists $F \subset V$, $\dim F = 3$, such that $H = \bigwedge^2 F$, or there exists a non-zero $v \in V$ such that $H = [v] \wedge V$. In the first case, $\dim H = 3$. In the second case, $\dim H = n$.

Proof. Let $\xi_1,\ \xi_2, \ldots$ be a base of H. The two-dimensional supports of ξ_1 and ξ_2 have a one dimensional intersection $[x] \subset V$ (if it was zero-dimensional $\xi_1 + \xi_2$ would not be decomposable, if it was two-dimensional, ξ_1 and ξ_2 would be proportional). We write $\xi_1 = x \wedge y_1$ and $\xi_2 = x \wedge y_2$. The support of ξ_3 must intersect both supports. If it is not in $[x]$, $[\xi_1, \xi_2, \xi_3] = \bigwedge^2 F$ where $F = [x, y_1, y_2]$ and there is no ξ_4 (the support of ξ_4 would cut F along a line. A bivector $\eta \in [\xi_1, \xi_2, \xi_3]$ whose support does not contain this line is such that $\xi_4 + \eta$ is not decomposable). If the intersection is $[x]$, $\xi_3 \in [x] \wedge V$. For $i > 3$, $\xi_i \in [x] \wedge V$ by the same arguments. QED

An invertible $\mathcal{R}$ satisfying (8.1), if $n \geq 4$, sends an n-dimensional space of decomposable bivectors of V on an n-dimensional space of decomposable bivectors of W. Doing so, it defines the map $\mathcal{R}_{\mathcal{P}}$ as announced. This map is bijective, the inverse being constructed from $\mathcal{R}^{-1}$, which satisfies condition (8.1) by Proposition 8.4.

The bijection $\mathcal{R}_{\mathcal{P}}$ sends projective lines on projective lines. To see this, observe that, u_1, u_2, v being non-zero vectors in V, $u_1 \wedge u_2 \neq 0$, the "point" $[v] \in \mathcal{P}(V)$ is on the "line" $[u_1, u_2]$ if and only if $u_1 \wedge u_2 \in [v] \wedge V$. By construction of $\mathcal{R}_{\mathcal{P}}$, the line $[u_1, u_2]$ is thus sent on the line corresponding to the decomposable bivector $\mathcal{R}(u_1 \wedge u_2)$.

By the fundamental theorem of projective geometry [Art], as $\dim V = \dim W \geq 3$, there exists an invertible linear map $\tilde{\mathcal{R}}_{\mathcal{P}} : V \to W$ which maps $\mathcal{P}(V)$ on $\mathcal{P}(W)$ as does $\mathcal{R}_{\mathcal{P}}$ (note that if we were working with $\mathbb{C}$ instead of $\mathbb{R}$, $\tilde{\mathcal{R}}_{\mathcal{P}}$ could be composed with the complex conjugation). The relation between $\mathcal{R}$ and $\tilde{\mathcal{R}}_{\mathcal{P}}$ is

$$\text{for any } (u,v) \in V^2, \quad \mathcal{R}(u \wedge v) \wedge \tilde{\mathcal{R}}_{\mathcal{P}}(u) = 0. \tag{8.4}$$

Or for any $(u,v) \in V^2$, $\mathcal{R}(u \wedge v)$ and $\tilde{\mathcal{R}}_{\mathcal{P}}(u) \wedge \tilde{\mathcal{R}}_{\mathcal{P}}(v)$ are proportional. Both term

define a linear map from $\bigwedge^2 V$ to $\bigwedge^2 W$. The proportionality factor between both maps must be a constant. With the notation introducing this section, $\mathcal{R} = \lambda(\tilde{\mathcal{R}}_{\mathcal{P}})^{\wedge 2}$, for some non-zero $\lambda \in \mathbb{R}$.

8.7. If $n = 3$, the two types of maximal spaces described in Lemma 8.6 have same dimension. They constitute two disconnected sub-varieties of the Grassmannian of 3-planes in $\bigwedge^2 V$. Thus if $\dim V = \dim W = n + 1 = 4$, either $\mathcal{R}$ sends the $[u] \wedge V$'s on the $[v] \wedge W$'s as in higher dimension, or it sends the $[u] \wedge V$'s on the $\bigwedge^2 F$'s (with $F \subset W$, $\dim F = 3$.) In the first case, the conclusion is the same as in the case $n \geq 4$. In the second case, we compose $\mathcal{R}$ with the bijection $\star : \bigwedge^2 W \to \bigwedge^2 W^*$, $\pi \mapsto \pi \rfloor \mu$, where $\mu \in \bigwedge^4 W^*$ is non-zero. The composed map $\star \circ \mathcal{R} : \bigwedge^2 V \to \bigwedge^2 W^*$ is in the first case. We have again the same conclusion for this map. This is the case (iv) in Theorem 8.1.

8.8. *Proposition.* Under the hypothesis of Theorem 8.1, if $\mathcal{R}$ is not bijective, there is a non-zero decomposable bivector in $(\mathrm{Im}\mathcal{R})^0$.

If E is a vector space and $F \subset E$ is a subspace we call $F^0 = \{\xi \in E^* |\text{for any } x \in F, \langle \xi, x\rangle = 0\}$. This space is called the annihilator and also denoted by $\mathrm{Ann}(F)$.

Proof of 8.8 when the dimension is even. We suppose $n+1 = \dim V = \dim W = 2m$. Let $\pi_1 \in \ker \mathcal{R}$. We can find $\pi_2, \ldots, \pi_m$ such that $\pi_1 \wedge \cdots \wedge \pi_m \neq 0$. For example, if $\mathrm{rk}\, \pi_1 = 2k$, we make $\pi_2 = \cdots = \pi_k = \pi_1$ and choose convenient decomposable bivectors $\pi_{k+1}, \ldots, \pi_m$. We see immediately from its definition that $\mathcal{R}^{(m)} = 0$. Thus $\mathrm{Im}\mathcal{R}$ is a subspace of W which does not contain any bivector of rank $2m$. Suppose it contains a bivector π of rank $2m - 2$. There exists $\xi \in W^*$ and $\eta \in W^*$ such that $\xi \wedge \eta \neq 0$ and $\xi \rfloor \pi = \eta \rfloor \pi = 0$. For any $\sigma \in \mathrm{Im}\mathcal{R}$ and any $x \in \mathbb{R}$, $(\pi + x\sigma) \wedge \cdots \wedge (\pi + x\sigma) = 0$. In particular, $\sigma \wedge \pi \wedge \cdots \wedge \pi = 0$. By contraction of ξ and η, we get $\langle \sigma, \xi \wedge \eta\rangle = 0$, using the hypothesis $\mathrm{rk}\pi = 2m - 2$. The decomposable bivector $\xi \wedge \eta$ is in $(\mathrm{Im}\mathcal{R})^0$. Now if the maximal rank for a bivector in $\mathrm{Im}\mathcal{R}$ is $2k < 2m - 2$, we choose π with this rank and $\pi_c \in W$ with rank $2m-2-2k$, such that $\pi + \pi_c$ has rank $2m-2$. With the previous argument, we conclude there is a decomposable bivector in $(\mathrm{Im}\mathcal{R} \oplus [\pi_c])^0$ and consequently in $(\mathrm{Im}\mathcal{R})^0$. QED

To treat the odd case we need two lemmas. Recall that if $\pi \in \bigwedge^q W$, then $\mathrm{supp}\pi$, the *support* of π, is determined as the intersection of the kernels of the $\xi \in W^*$ such that $\xi \rfloor \pi = 0$. For any positive integer m, we set $\pi^{\wedge m} = \pi \wedge \cdots \wedge \pi$ (repeated m times).

8.9. *Lemma.* If $\pi \in \bigwedge^q W$, $\mathrm{supp}\pi^{\wedge m} \subset \mathrm{supp}\pi^{\wedge p}$ for $1 \leq p \leq m$. If $\pi \in \bigwedge^2 W$, with rank $2m$, $\mathrm{supp}\pi = \mathrm{supp}\pi^{\wedge p}$ for $1 \leq p \leq m$.

Proof. $\xi \rfloor \pi^{\wedge m} = m(\xi \rfloor \pi) \wedge \pi^{\wedge m-1}$ is proportional to $(\xi \rfloor \pi^{\wedge p}) \wedge \pi^{\wedge m-p} = p(\xi \rfloor \pi) \wedge \pi^{\wedge m-1}$, so if $\xi \rfloor \pi^{\wedge p} = 0$ then $\xi \rfloor \pi^{\wedge m} = 0$. For the second statement we use $\pi = e_1 \wedge e_2 + \cdots + e_{2m-1} \wedge e_{2m}$ and $\pi^{\wedge m} = e_1 \wedge \cdots \wedge e_{2m}$. QED

8.10. *Lemma.* Let $\mu \in \bigwedge^q W$, $\mu \neq 0$ and $\phi \in W$. If $\phi \in \mathrm{supp}\mu$, then $\mathrm{supp}(\phi \wedge \mu) \subset \mathrm{supp}\mu$. If $\phi \notin \mathrm{supp}\mu$, then $\mathrm{supp}(\phi \wedge \mu) = \mathrm{supp}\mu \oplus [\phi]$.

Proof. The first statement is easy. For the second, consider a $\xi \in W^*$ with

$\xi\rfloor(\phi\wedge\mu) = 0$. Then $\langle\xi,\phi\rangle\mu = \phi\wedge(\xi\rfloor\mu)$. But there exists an $\eta \in W^*$ with $\eta\rfloor\mu = 0$ and $\langle\phi,\eta\rangle \neq 0$. Contracting η in the previous relation gives $\langle\phi,\eta\rangle\xi\rfloor\mu = 0$. Thus $\xi\rfloor\mu = 0$ and using again the same relation $\langle\xi,\phi\rangle = 0$. So ξ vanishes on $\mathrm{supp}\mu \oplus [\phi]$. QED

Proof of 8.8 when the dimension is odd. We suppose $n+1 = \dim V = \dim W = 2m+1$. If there is a non-decomposable bivector $\pi_1 \in \ker\mathcal{R}$, let $(e_0,\ldots,e_n)$ be a base of V such that $\pi_1 = e_0 \wedge e_1 + e_2 \wedge e_3 + \cdots$. It is easy to see that the elements of the standard base of $\bigwedge^n V$ all have the form $\pi_1 \wedge \pi_2 \wedge \cdots \wedge \pi_m$, with, for $i \geq 2$, $\pi_i = e_{i_1} \wedge e_{i_2}$, $0 \leq i_1 < i_2 \leq n$. For example to get $e_1 \wedge \cdots \wedge e_n$, we take $\pi_2 = e_1 \wedge e_4$, $\pi_3 = e_5 \wedge e_6$, etc. Note that this construction works because $\mathrm{rk}\,\pi_1 \geq 4$. From this and the fundamental property of $\mathcal{R}^{(m)}$ stated in Lemma 8.3, we deduce that $\mathcal{R}^{(m)} \equiv 0$. So the bivectors in $\mathrm{Im}\mathcal{R}$ have rank at most $2m-2$. By the same argument as in the even case, there is a decomposable bivector in $(\mathrm{Im}\mathcal{R})^0$.

If there are only decomposable bivectors in $\ker\mathcal{R}$ we can deduce by the same arguments that $\mathrm{rk}\mathcal{R}^{(m)} \leq 2$. If $\mathrm{rk}\mathcal{R}^{(m)} = 0$, we conclude with the above argument. If $\mathrm{rk}\mathcal{R}^{(m)} = 1$, there is a $\pi \in \mathrm{Im}\mathcal{R}$ with $\pi^{\wedge m} \neq 0$, and a non-zero $\xi \in W^*$ such that $\xi\rfloor\sigma^{\wedge m} = 0$ for all $\sigma \in \mathrm{Im}\mathcal{R}$. For any such σ we have $\xi\rfloor(\sigma \wedge \pi^{\wedge m-1}) = 0$. Then $(\xi\rfloor\sigma) \wedge \pi^{\wedge m-1} = 0$ and $\xi\rfloor\sigma \in \mathrm{supp}\pi^{\wedge m-1} = \mathrm{supp}\pi$ by 8.9 and 8.10 Then $\xi\rfloor\sigma = \zeta\rfloor\pi$ for some $\zeta \in W^*$ and $(\zeta\rfloor\pi) \wedge \pi^{\wedge m-1} = 0$. Thus $\zeta\rfloor\pi^{\wedge m} = 0$ and $\zeta\rfloor\pi = 0$ (same support). Finally $\xi\rfloor\sigma = 0$ for all $\sigma \in \mathrm{Im}\mathcal{R}$ which gives many decomposable bivectors in $(\mathrm{Im}\mathcal{R})^0$.

The last case is $\mathrm{rk}\mathcal{R}^{(m)} = 2$. There is a $\pi \in \mathrm{Im}\mathcal{R}$ with $\pi^{\wedge m} \neq 0$, a non-zero $\xi \in W^*$ with $\xi\rfloor\pi = 0$, and an $\eta \in W^*$ such that $\xi \wedge \eta \neq 0$ and $\xi \wedge \eta\rfloor\sigma^{\wedge m} = 0$ for any $\sigma \in \mathrm{Im}\mathcal{R}$. Then $\xi \wedge \eta\rfloor(\sigma \wedge \pi^{\wedge m-1}) = 0$ for any such σ. Then $\eta\rfloor\big((\xi\rfloor\sigma) \wedge \pi^{\wedge m-1}\big) = 0$. Then the support of the parenthesis is strictly included in W, so $\xi\rfloor\sigma \in \mathrm{supp}\pi^{\wedge m-1}$ by Lemma 8.10. Then $\xi\rfloor\sigma = \zeta\rfloor\pi$ for some ζ and the relation becomes $(\eta \wedge \zeta)\rfloor\pi^{\wedge m} = 0$. If $\mu \in \bigwedge^{2m+1} W$ is non-zero, this means $(\eta \wedge \zeta \wedge \xi)\rfloor\mu = 0$ or $\zeta = a\xi + b\eta$. Finally $\xi\rfloor\sigma = b\eta\rfloor\pi$ and $\langle\xi \wedge \eta, \sigma\rangle = 0$ for any $\sigma \in \mathrm{Im}\mathcal{R}$. QED

8.11. *Proposition.* If there is a non-zero decomposable bivector $\xi\wedge\eta \in (\mathrm{Im}\mathcal{R})^0 \subset \bigwedge^2 W^*$ but no non-zero element $\zeta \in W^*$ such that, for all $\pi \in \bigwedge^2 V$, $\zeta\rfloor\mathcal{R}(\pi) = 0$, then there is a non-zero $\phi \in W$ such that $\mathcal{R}(\pi) \wedge \phi = 0$ for all $\pi \in \bigwedge^2 V$.

Proof. We deduce from (8.1) that for all $(u,v) \in V^2$,

$$0 = \xi \wedge \eta\rfloor\mathcal{R}(u \wedge v) \wedge \mathcal{R}(u \wedge v) = \big(\xi\rfloor\mathcal{R}(u \wedge v)\big) \wedge \big(\eta\rfloor\mathcal{R}(u \wedge v)\big). \tag{8.5}$$

This is the collinearity of two elements of W. To see how this collinearity may occur, we take the value of the above expression on $(\alpha,\beta) \in (W^*)^2$. We get

$$\langle\mathcal{R}(u \wedge v), \xi \wedge \alpha\rangle\langle\mathcal{R}(u \wedge v), \eta \wedge \beta\rangle = \langle\mathcal{R}(u \wedge v), \xi \wedge \beta\rangle\langle\mathcal{R}(u \wedge v), \eta \wedge \alpha\rangle.$$

The relation says that a polynomial in the coordinates of the vectors u, v, α, β possesses two factorizations. As the degrees are the same, if the non-zero polynomial $\langle\mathcal{R}(u\wedge v), \xi\wedge\beta\rangle$ divides the polynomial $\langle\mathcal{R}(u\wedge v), \eta\wedge\beta\rangle$, the quotient

is a real number λ. Then $(\eta - \lambda\xi)\rfloor\mathcal{R}(u \wedge v) = 0$ for all $(u, v) \in V^2$, which is excluded by the hypothesis.

But $\langle \mathcal{R}(u \wedge v), \xi \wedge \beta\rangle$ cannot divide the other factor. So it cannot be irreducible as a polynomial in the coordinates of u, v, β. We claim that the factorization is

$$\langle \mathcal{R}(u \wedge v), \xi \wedge \beta\rangle = \langle \omega, u \wedge v\rangle\langle \phi, \beta\rangle, \quad \text{with} \quad \omega \in \bigwedge^2 V^*, \quad \phi \in W. \tag{8.6}$$

The proof is as follows. The total degree in the coordinates of β is one for the product, so it should be one for a factor, zero for the other factor. As the product is homogeneous, the factors are homogeneous. The same is true for the coordinates of u and of v. So the factorization is: linear times bilinear. But u and v cannot be separated, because $\mathcal{R}$ is antisymmetric in (u, v). So there exist ω and ϕ as claimed. We have $\xi\rfloor\mathcal{R}(u \wedge v) = \langle \omega, u \wedge v\rangle\phi$. We have assumed that this quantity is not identically zero. Putting it in (8.5) we find $\big(\eta\rfloor\mathcal{R}(u \wedge v)\big) \wedge \phi = 0$. From (8.6) we deduce that any $\beta \in \ker\phi$ is such that $\xi \wedge \beta \in (\mathrm{Im}\mathcal{R})^0$. So we have Relation (8.5) with β instead of η, and we deduce $\big(\beta\rfloor\mathcal{R}(u \wedge v)\big) \wedge \phi = 0$. Finally $\mathcal{R}(u \wedge v) \wedge \phi = 0$. QED

9. From decomposable bivectors to decomposable 2-forms

Theorem 8.1 gives enough results on an $\mathcal{R} : \bigwedge^2 V \to \bigwedge^2 W$ satisfying (8.1) to deduce more complete results on the particular case where $W = V^*$, and where $\mathcal{R}$ is associated to a quadrilinear form in $\mathcal{P}^{2,2}_{\mathcal{A}}V^*$, i.e. a quadrilinear form with the symmetries of the Riemannian curvature tensor (see 6.4 and 5.5). Let $R_{\mathcal{A}} : V^4 \to \mathbb{R}$, $(u, v, w, x) \mapsto R_{\mathcal{A}}(u, v; w, x)$ be such a form. It satisfies $R_{\mathcal{A}}(u, v; w, x) = -R_{\mathcal{A}}(v, u; w, x) = -R_{\mathcal{A}}(u, v; x, w)$ and the algebraic Bianchi identity $R_{\mathcal{A}}(u, v; w, x) + R_{\mathcal{A}}(v, w; u, x) + R_{\mathcal{A}}(w, u; v, x) = 0$. It defines a linear map $\mathcal{R} : \bigwedge^2 V \to \bigwedge^2 V^*$. This map is symmetric: $\langle \mathcal{R}(u \wedge v), w \wedge x\rangle = \langle \mathcal{R}(w \wedge x), u \wedge v\rangle$. We write $\mathcal{R}(u \wedge v) = R_{\mathcal{A}}(u, v; ., .)$. We define: $\ker R_{\mathcal{A}} = \{u \in V, R_{\mathcal{A}}(u, .; ., .) = 0\}$. We write Condition (8.1)

$$R_{\mathcal{A}}(u, v; ., .) \wedge R_{\mathcal{A}}(u, v; ., .) = 0 \quad \text{for any } (u, v) \in V^2. \tag{9.1}$$

9.1. *Proposition.* Let V be a real vector space, $R_{\mathcal{A}} \in \mathcal{P}^{2,2}_{\mathcal{A}}V^*$. If $R_{\mathcal{A}}$ satisfies (9.1), then the induced quadrilinear form $\hat{R}_{\mathcal{A}} : (V/\ker R_{\mathcal{A}})^4 \to \mathbb{R}$ satisfies (9.1).

9.2. *Theorem.* Let V be a real vector space, $n + 1 = \dim V \geq 3$. Let $R_{\mathcal{A}} \in \mathcal{P}^{2,2}_{\mathcal{A}}V^*$ and $\mathcal{R} : \bigwedge^2 V \to \bigwedge^2 V^*$ associated to it. Suppose $R_{\mathcal{A}}$ satisfies (9.1) and $\ker R_{\mathcal{A}} = \{0\}$. Then either (i) there exist an invertible symmetric linear map $B : V \to V^*$ and $\varepsilon = \pm 1$ such that $\mathcal{R} = \varepsilon B^{\wedge 2}$, or (ii) there exist a non-zero $\phi \in V^*$ and a non-degenerate quadratic form g defined on $\ker\phi$ such that $R_{\mathcal{A}}(u, v; u, v) = g\big(\phi\rfloor(u \wedge v)\big)$.

9.3. *Remark.* The second member of the last formula is well-defined: $\phi\rfloor(u \wedge v) = \phi(u)v - \phi(v)u \in \ker\phi$. The formula uniquely defines $R_{\mathcal{A}}$ according to Proposition 5.4. We will need the complete formula for $R_{\mathcal{A}}$. Let $\tilde{g}$ be the bilinear form associated to g. The complete formula is $R_{\mathcal{A}}(u, v; w, x) = \tilde{g}\big(\phi\rfloor(u \wedge$

$v), \phi\rfloor(w\wedge x)\big)$, simply because this formula satisfies the algebraic Bianchi identity, as easily seen using an arbitrary extension of g to V.

Proof of 9.1. The support of $R_{\mathcal{A}}$, seen as a tensor of $\bigotimes^4 V^*$, is $(\ker R_{\mathcal{A}})^0$. The support of $R_{\mathcal{A}}(u,v;.,.) \in \bigwedge^2 V^*$ is included in $(\ker R_{\mathcal{A}})^0$. If $R_{\mathcal{A}}(u,v;.,.)$ is decomposable as an element of $\bigwedge^2 V^*$ it is decomposable as an element of $\bigwedge^2 (\ker R_{\mathcal{A}})^0$. QED

Proof of 9.2. We begin with the case where $\mathcal{R}$ is invertible and $n \geq 4$. By Theorem 8.1, there exists a linear map $B : V \to V^*$ and an $\varepsilon = \pm 1$ such that $\mathcal{R} = \varepsilon B^{\wedge 2}$. We choose $\varepsilon = 1$. The other choice of sign is similar. Let b be the bilinear form associated to B, i.e. $b(u,.) = B(u)$. Then $R_{\mathcal{A}}(u,v;w,x) = b(u,w)b(v,x) - b(u,x)b(v,w)$. We write

$$R_{\mathcal{A}}(u,v;w,x)+R_{\mathcal{A}}(v,w;u,x)+R_{\mathcal{A}}(w,u;v,x) = b(u,x)\big(b(w,v)-b(v,w)\big)+\text{perm.}$$

By the Bianchi identity this expression vanishes. It vanishes for any $x \in V$ so

$$\big(b(w,v)-b(v,w)\big)B(u) + \big(b(u,w)-b(w,u)\big)B(v) + \big(b(v,u)-b(u,v)\big)B(w) = 0.$$

We claim that $b(w,v) = b(v,w)$ for any (v,w) with $v \wedge w \neq 0$, and thus b is symmetric. We take u such that $u\wedge v\wedge w \neq 0$. By invertibility of B, $B(u)$, $B(v)$ and $B(w)$ are independent. The identity appears as a zero linear combination of these three covectors. So the coefficients are zero.

We continue with an invertible $\mathcal{R}$, considering now $n = 3$. By 8.7, the situation is the same as if $n \geq 4$, except for the case where $\mathcal{R}$ is the composition of an $\mathcal{R}' : \bigwedge^2 V \to \bigwedge^2 V$ with the isomorphism $\star : \bigwedge^2 V \to \bigwedge^2 V^*$. In this case there is a $C : V \to V$ such that $\mathcal{R}' = \pm C^{\wedge 2}$. We claim that such a composition does not give an $R_{\mathcal{A}}$ which satisfies the Bianchi identity. Identifying $\bigwedge^4 V$ with $\mathbb{R}$ using $\star$, we write $R_{\mathcal{A}}(u,v,w,x) = u \wedge v \wedge C(w) \wedge C(x)$. The Bianchi identity is

$$u \wedge v \wedge C(w) + u \wedge C(v) \wedge w + C(u) \wedge v \wedge w = 0.$$

Writing this identity on base vectors e_0, e_1, e_2, e_3 we deduce easily that the matrix of C is zero. So this case is excluded, and the proof is complete in the case of an invertible $\mathcal{R}$.

We suppose $\mathcal{R}$ is not invertible. As $\ker R_{\mathcal{A}} = \{0\}$ excludes Case (ii) of Theorem 8.1, we are in Case (i). There is a non-zero $\phi \in V^*$ such that $R_{\mathcal{A}}(u,v;.,.)\wedge\phi = 0$ for all $(u,v) \in V^2$. It implies $R_{\mathcal{A}}(u,v;w,x) = 0$ for any $(w,x) \in (\ker\phi)^2$.

An $R_{\mathcal{A}} \in \mathcal{P}^{2,2}_{\mathcal{A}}(V)$ associates to a $u \in V$ the quadratic form $v \mapsto R_{\mathcal{A}}(u,v;u,v)$. In the present case, we have for any $w \in \ker\phi$, $R_{\mathcal{A}}(u+w,v;u+w,v) = R_{\mathcal{A}}(u,v;u,v)$, provided $v \in \ker\phi$. Then any $u \in V$ such that $\langle\phi,u\rangle = 1$ defines the same quadratic form g on $\ker\phi$. If $\langle\phi,u\rangle \neq 1$, the quadratic form defined by u on $\ker\phi$ is still proportional to g. If g was degenerate, we would have a non-zero $v \in \ker\phi$ such that $R_{\mathcal{A}}(u,v;u,.) = 0$ for any u. By Lemma 5.4 this would imply $R_{\mathcal{A}}(.,v;.,.) = 0$, which would contradict $\ker R_{\mathcal{A}} = \{0\}$. The last statement to prove is the formula for $R_{\mathcal{A}}$. We have

$$g\big(\phi\rfloor(u\wedge v)\big) = R_{\mathcal{A}}\big(w, \phi\rfloor(u\wedge v); w, \phi\rfloor(u\wedge v)\big) \quad \text{where } w \in V \text{ satisfies } \langle\phi,w\rangle = 1.$$

Suppose $\langle \phi, u\rangle \neq 0$ and set $w = u/\langle \phi, u\rangle$. The second member is $R_{\mathcal{A}}(u, v; u, v)$ as claimed. If $\langle \phi, u\rangle = 0$ we rather set $w = v/\langle \phi, v\rangle$ to get the same result. If $\langle \phi, u\rangle = \langle \phi, v\rangle = 0$ the equality in 9.2 is $0 = 0$. QED

The description in Theorem 9.2 is not completely satisfying because we do not see if one can pass continuously from Case (i) to Case (ii). For information, we give without proof the following result.

9.4. *Proposition.* Let V be a real vector space, $\dim V = n+1 \geq 3$. We choose a non-zero element of $\bigwedge^{n+1} V^*$ and define the isomorphism $\star : \bigwedge^2 V \to \bigwedge^{n-1} V^*$ associated to it. Let $G : V^* \to V$, satisfying ${}^tG = G$. We construct an $\mathcal{R} : \bigwedge^2 V \to \bigwedge^2 V^*$ conjugating by $\star$ the map $G^{\wedge n-1} : \bigwedge^{n-1} V^* \to \bigwedge^{n-1} V$. We set $R_{\mathcal{A}}(u, v; w, x) = \langle \mathcal{R}(u \wedge v), w \wedge x\rangle$. Then $R_{\mathcal{A}} \in \mathcal{P}^{2,2}_{\mathcal{A}}(V)$ and satisfies (9.1). We have $\ker R_{\mathcal{A}} = \{0\}$ if and only if $\mathrm{rk} G \geq n$. If G has rank $n+1$, we are in Case (i) of Theorem 9.2, with $B = G^{-1}$. If G has rank n, we are in Case (ii), with $[\phi] = \ker G$.

10. Free motion with quadratic pre-Lagrangian

Using the results of Section 9 we can solve the problem raised at the end of Section 7. We first derive the equation for a compatible pair $(R, \mathcal{H})$. The biquadratic polynomial $R : V^2 \to \mathbb{R}$, $(q, v) \mapsto R(q, v)$ defines a quadratic form on the tangent space $T_q\mathcal{H}$ at any point $q \in \mathcal{H}$. The compatibility condition is simply that this form is invariant by parallel transport. The parallel transport of a vector $w \in T_q\mathcal{H}$ along the curve with tangent velocity $\dot{q} \in T_q\mathcal{H}$ is defined by the equation $\dot{w} = \lambda q$, where λ is a real function. We use the antisymmetric polarization $R_{\mathcal{A}}$ of R, which is such that $R(q, w) = R_{\mathcal{A}}(q, w; q, w)$. We write

$$0 = \frac{d}{dt} R(q, w) = \frac{d}{dt} R_{\mathcal{A}}(q, w; q, w) = 2R_{\mathcal{A}}(q, w; \dot{q}, w).$$

We arrive at a first form of the compatibility condition:

$$\text{for any } q \in \mathcal{H}, u \in T_q\mathcal{H}, w \in T_q\mathcal{H}, \quad R_{\mathcal{A}}(q, w; u, w) = 0. \tag{10.1}$$

According to 5.9 we can apply 5.4 to the case of the trilinear form $R_{\mathcal{A}}(q, .; ., .)$. Thus (10.1) is equivalent to

$$\text{for any } q \in \mathcal{H}, u \in T_q\mathcal{H}, v \in T_q\mathcal{H}, w \in T_q\mathcal{H}, \quad R_{\mathcal{A}}(q, v; u, w) = 0. \tag{10.2}$$

The last expression depends on q and v through the bivector $q \wedge v$. We can change the hypothesis $v \in T_q\mathcal{H}$ into $v \in V$. Thinking of $R_{\mathcal{A}}(q, v; ., .)$ as a 2-form in the missing arguments, and dh as the 1-form, differential of the screen function h, the compatibility condition (10.2) reads

$$\text{for any } q \in \mathcal{H}, v \in V, \quad dh|_q \wedge R_{\mathcal{A}}(q, v; ., .) = 0. \tag{10.3}$$

In particular, $R_{\mathcal{A}}$ satisfies (9.1). The two following results correspond to Proposition 9.1.

10.1. *Lemma.* Let $(R_{\mathcal{A}}, \mathcal{H})$ be a compatible pair. Let $h(q) = 1$ be the equation of the screen $\mathcal{H}$. If $k \in \ker R_{\mathcal{A}}$, then $\langle dh|_q, k\rangle = 0$ for any $q \in \mathcal{H}$.

Proof. If $q \notin \ker R_{\mathcal{A}}$, there exists a $v \in V$ such that $R_{\mathcal{A}}(q, v; ., .) \neq 0$. If $q \in \mathcal{H}$, we choose (10.3) as the compatibility equation and contract $k \in \ker R_{\mathcal{A}}$. It gives $\langle dh|_q, k\rangle = 0$. Suppose $q \in \mathcal{H} \cap \ker R_{\mathcal{A}}$. By continuity we should have $\langle dh|_q, k\rangle = 0$, and in particular $\langle dh|_q, q\rangle = h(q) = 0$. This is impossible, so $\mathcal{H} \cap \ker R_{\mathcal{A}} = \emptyset$. QED.

10.2. *Proposition.* Let $R_{\mathcal{A}} \in \mathcal{P}^{2,2}_{\mathcal{A}}(V)$ not equal to zero, $R^0_{\mathcal{A}} \in \mathcal{P}^{2,2}_{\mathcal{A}}(V/\ker R_{\mathcal{A}})$ be the quadrilinear form induced on the quotient space $V/\ker R_{\mathcal{A}}$. Then any screen $\mathcal{H}^0$ of $V/\ker R_{\mathcal{A}}$ compatible with $R^0_{\mathcal{A}}$ gives by pull-back a "cylindric" screen $\mathcal{H}$ compatible with $R_{\mathcal{A}}$. Reciprocally every screen compatible with $R_{\mathcal{A}}$ is contained in such a cylindric screen.

Proof. We have $T_{q^0}\mathcal{H}^0 = T_q\mathcal{H}/\ker R_{\mathcal{A}}$ for any q projecting on q^0 by the canonical projection $V \to V/\ker R_{\mathcal{A}}$. Denote by v^0 and w^0 the respective canonical projections of v and w. We have $R^0_{\mathcal{A}}(q^0, v^0; w^0, w^0) = R_{\mathcal{A}}(q, v; w, w)$. If the compatibility condition (10.1) is satisfied for $(R^0_{\mathcal{A}}, \mathcal{H}^0)$, it is satisfied for $(R_{\mathcal{A}}, \mathcal{H})$, and reciprocally. QED

10.3. *Proposition.* Suppose $(R_{\mathcal{A}}, \mathcal{H})$ is a compatible pair with $\ker R_{\mathcal{A}} = \{0\}$. Then either (i) there exists a non-degenerate quadratic form g on V and a $\lambda \in \mathbb{R}$ such that the screen $\mathcal{H}$ is included in the quadric of equation $g(q) = 1$ and $R_{\mathcal{A}}(q, v; q, v) = \lambda g(v)$ for any $q \in \mathcal{H}$ and $v \in T_q\mathcal{H}$, or (ii) there exists a non-zero $\phi \in V^*$, a non-degenerate quadratic form g defined on $\ker\phi$ and a $\lambda \in \mathbb{R}$ such that the screen $\mathcal{H}$ is included in the hyperplane of equation $\langle\phi, q\rangle = 1$ and $R_{\mathcal{A}}(q, v; q, v) = \lambda g(v)$ for any $q \in \mathcal{H}$ and $v \in T_q\mathcal{H} = \ker\phi$, or (iii) $n = 1$ i.e. $\dim V = 2$.

Proof. As (10.3) implies (9.1), we are in case (i) or (ii) of Theorem 9.2. In case (i) Equation (10.3) becomes $dh|_q \wedge B(q) \wedge B(v) = 0$. This is true for any v so if $\dim V \geq 3$ we have $dh|_q \wedge B(q) = 0$ for any $q \in \mathcal{H}$. We set $g(q) = \langle B(q), q\rangle$. The equation is $dh|_q \wedge dg|_q = 0$. The screen being a level hypersurface of the screen function h, it coincides according to the equation with a level hypersurface of g. We compute $R_{\mathcal{A}}(q, v; q, v) = \langle \mathcal{R}(q \wedge v), q \wedge v\rangle = \varepsilon\big(\langle B(q), q\rangle\langle B(v), v\rangle - \langle B(q), v\rangle^2\big) = \varepsilon g(q)g(v)$. After a convenient rescaling this gives the first case. In case (ii) of Theorem 9.2, according to 9.3, $\mathcal{R}_{\mathcal{A}}(q, v; ., .) = \phi \wedge \tilde{g}\big(\phi\rfloor(q \wedge v), .\big)$, where the second factor is only defined up to the addition of a scalar multiple of ϕ, but the result of the wedge product is well defined. Equation (10.3) is true in particular for any $v \in \ker\phi$ and it gives $\langle\phi, q\rangle dh|_q \wedge \phi \wedge \tilde{g}(v, .) = 0$ for any $q \in \mathcal{H}$. Thus we have $dh|_q \wedge \phi \wedge \xi = 0$ for any $\xi \in V^*$, which implies, if $\dim V \geq 3$, $dh|_q \wedge \phi = 0$ for any $q \in \mathcal{H}$. This proves that $\mathcal{H}$ is flat. A rescaling gives the formulas as written. QED

11. Conclusion

It is time to explain how the various statements we established may be used. Suppose we wish to study a system (2.1) on a screen $\mathcal{H}$. If there is a non-trivial algebra of first integrals, we must determine it, and as we explained at the

beginning of Section 4, it is reasonable to limit ourselves to the first integrals which are polynomial in the velocities.

Laplace [Lap] gave what is maybe the first example of determination of the algebra of the first integrals. The system he considered is the Kepler problem in space. He devised a systematic research of polynomial first integrals, and developed it until he found all the classical first integrals. They are at most quadratic in the velocities. Actually Laplace already knew the first integrals he found. They all appear in an earlier work by Lagrange [Lag], including the so-called Laplace-Runge-Lenz vector. But the method is interesting and general. The first step is to determine an *a priori* information on the higher degree term of the integral (compare for example [Whi], p. 332).

In Section 6 we put this information in a simple and general form: after homogenization, the higher order term is a multilinear form, whose symmetry corresponds to a rectangular Young tableau with two rows. In other words it is a polynomial in $q \wedge v$, the projective impulsion.

Suppose we find a quadratic first integral, i.e. a function $G = T - U$, where U is a function of the position only, T a function of the position and the velocity which is homogeneous polynomial of degree 2 in the velocity. Such a first integral has a lot to do with a Hamiltonian, as emphasized in the case of a flat $\mathcal{H}$ by Lundmark. Translated into the language of projective dynamics, Lundmark's remark is simply: if G is not the Hamiltonian, it may become the Hamiltonian after a change of screen.

We proposed a simple test to decide if a screen exists that makes G the Hamiltonian. The condition we found is almost exactly the characterization of Lundmark's cofactor systems. Projective dynamics simplifies the statements, gives a geometrical interpretation and brings to these theories the powerful tools from Young theory (compare [Ben]).

The test is as follows. We must decide if $L = T + U$ is a Lagrangian for some screen. Following Proposition 7.8, we just look at T. We homogenize T in an $R_{\mathcal{A}} \in \mathcal{P}^{2,2}_{\mathcal{A}}(V)$, i.e. in a quadrilinear form having the same symmetries as the Riemann curvature tensor in Riemannian geometry. This symmetry corresponds to a 2×2 Young tableau.

The quadrilinear form $R_{\mathcal{A}}$ does not depend on the screen. Guided by Proposition 10.2 we ask if $R_{\mathcal{A}}$ has a non-trivial kernel. In this case G is not a Hamiltonian, but according to Proposition 7.5 it can still give a preserved pre-symplectic form to the system. This form exists if the reduced pair $(R^0_{\mathcal{A}}, \mathcal{H}^0)$ in Proposition 10.2 is a compatible pair, which we decide using Proposition 10.3.

If the kernel of $R_{\mathcal{A}}$ is trivial, we use Proposition 10.3 to see if there is a screen such that G is a Hamiltonian, and which is the screen. The screen must be contained in a hyperplane or a quadric. Proposition 10.3 provides a justification for the definition of a cofactor system.

The theory continues asking if there is another first integral, if it is a Hamiltonian for some screen, if with the first it forms a "cofactor pair". This would imply bi-

hamiltonianity, separability and integrability. Thus for a system with n degrees of freedom, two quadratic first integrals may be enough to integrate. All this is explained in the thesis of Lundmark.

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